A model of two-velocity particles filtration with variable injection

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Abstract. The filtration problems are actual in the design of underground storage facilities for hazardous waste. When the grout is injected into the porous soil, the fluid penetrates deep into the rock and, when solidified, blocks the pores. A model of suspension flow with 2-size particles moving with different velocities in a porous medium is considered. The proposed model of deep bed filtration generalizes the known equations of mass balance and particle capture kinetics for a fluid flow with various particles. The injection of a suspension with periodically changing concentration is considered. Exact and asymptotic solutions are obtained. The asymptotics rapidly converges to the numerical solution.

1 Introduction

Filtration is one of the most important problems in the design of underground and hydraulic structures [1-3]. Underground storage of toxic and radioactive waste must be reliably isolated from groundwater and reservoir waters. One of the modern ways to construct the waterproof walls in the ground is to inject under pressure a cement-based suspension. The grout is filtered in the porous soil, the concrete grains penetrate into the fine pores of the rock and block them. After the fluid solidifies, the rock becomes firm and water resistant.

The long-term transport of particle suspensions in porous media with particle capture throughout the entire length of the porous rock is called deep bed filtration. There are different mathematical models of deep bed filtration: the classical model for a mono dispersed suspension flow in a porous medium includes the equations of the particle balance and of the capture kinetics [4]; the population balance models for a poly dispersed suspension motion in a porous medium [5–8], which take into account the size distribution of pores and particles.

All these models assume that the suspended particles are transported with same velocity. However, experimental results have shown that the particle velocities depend on their size [9]. Suspension in the porous medium moves unevenly due to different pore sizes. With increasing the pores diameter the velocity of a fluid with suspended particles becomes greater. Large particles moving only through large pores, so their average velocity is greater than particles of small particles that are transported both over large and small pores.

The fluid flow with two types of solid particles moving with different velocities in a porous medium is considered. A mathematical model which generalizes the known

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equations of mass balance and particle capture kinetics for a flow of fluid with various particles is developed in [10]. When the grout is injected, the concentration of suspended particles in the fluid can vary. The periodic changes of the concentrations at the porous medium inlet are considered in the paper. The filtration problem with variable coefficients does not have an exact solution, so the methods of asymptotic and numerical modeling are used. The asymptotic solution of the problem with a small limit deposit is constructed by methods [11, 12]. The asymptotics rapidly converges to the numerical solution.

The mathematical model of 2-size particles transport is presented in Section 2. Section 3 provides the exact solution for constant coefficients. The asymptotic solution for a small limit deposit is obtained in Section 4. Section 5 is devoted to the comparison of the asymptotics with the numerical solution. The conclusion in Section 6 finalizes the paper.

2 Mathematical model

Consider quasi-linear hyperbolic system of equations in the domain $\Omega = \{0 < x < 1, t > 0\}$

$$\frac{\partial(C+S)}{\partial t} + \alpha \frac{\partial C}{\partial x} = 0; \quad (1)$$

$$\frac{\partial S}{\partial t} = \Lambda(S)C, \quad (2)$$

where $C(x, t)$, $S(x, t)$ are the suspended and retained particles concentrations; α is the velocity of the suspended particles; $\Lambda(S)$ is the filtration coefficient.

The boundary conditions for the system (1), (2) are set at the inlet of the porous medium $x = 0$ and at the initial moment $t = 0$:

$$C(x, t)|_{x=0} = 1; \quad (3)$$

$$C(x, t)|_{t=0} = 0; \quad (4)$$

$$S(x, t)|_{t=0} = 0. \quad (5)$$

The solution of the system (1)-(5) is considered in [13].

Now consider the model of 2-velocity particles transport. Let the suspension flow has particles of two different sizes d_1 and d_2 with velocities α_1 and α_2 respectively. The filter has pores of two diameters D_1 , D_2 and $D_1 < d_1 < D_2$, $D_1 < d_2 < D_2$. Thus, both types of particles get stuck in small pores with diameter D_1 and pass freely through large pores with diameter D_2 . The transport of particles of each type is described by a mass balance equation of type (1) in the dimensionless coordinates linking the concentrations of suspended $C_1(x, t)$, $C_2(x, t)$ and retained particles $S_1(x, t)$, $S_2(x, t)$. The particles of both types compete for the small pores D_1 . The growth rates of the deposits $\partial S_1 / \partial t$ and $\partial S_2 / \partial t$ are proportional to the concentrations of suspended particles C_1 and C_2 . The filtration coefficients $\Lambda_1(S)$ and $\Lambda_2(S)$ depend on the amount of the current deposit $S(x, t)$ consisting of both types of retained particles S_1 and S_2 : $S = S_1 + S_2$.

In the domain Ω the concentrations of suspended and retained particles satisfy the quasi-linear hyperbolic system

$$\frac{\partial(C_i + S_i)}{\partial t} + \alpha_i \frac{\partial C_i}{\partial x} = 0; \quad (6)$$

$$\frac{\partial S_i}{\partial t} = \Lambda_i(S)C_i, \quad i = 1, 2, \quad (7)$$

with the boundary and initial conditions similar to (3)-(5):

$$C_i(x, t)|_{x=0} = f_i(t), \quad i = 1, 2; \quad (8)$$

$$C_i(x, t)|_{t=0} = 0, \quad i = 1, 2; \quad (9)$$

$$S_i(x, t)|_{t=0} = 0, \quad i = 1, 2. \quad (10)$$

At $t = 0$ there are no suspended and retained particles in the porous medium (the initial conditions 9, 10). The suspension with variable concentrations of 2-size particles is injected into the filter inlet $x = 0$ (the boundary condition 8). The particles are being transported towards the filter outlet $x = 1$ at different velocities α_1 and α_2 and gradually fill the entire porous medium. The mobile boundary of the domain which contains particles of a certain size is called a concentration front. The concentration fronts Γ_1 and Γ_2 are the characteristics defined by the equations $t = x/\alpha_1$ and $t = x/\alpha_2$ respectively.

Let $0 < \alpha_2 < \alpha_1$. The domain Ω is divided into 3 subdomains by the characteristics $t = x/\alpha_1$ and $t = x/\alpha_2$:

$$\Omega_0 = \left\{ 0 < x < 1, 0 < t < \frac{x}{\alpha_1} \right\}, \Omega_1 = \left\{ 0 < x < 1, \frac{x}{\alpha_1} < t < \frac{x}{\alpha_2} \right\}, \Omega_2 = \left\{ 0 < x < 1, t > \frac{x}{\alpha_2} \right\}.$$

In the domain Ω_0 the porous medium is empty; in the domain Ω_1 only fast particles of type 1 are presence; in the domain Ω_2 there are particles of both types.

To determine the solution in the domains Ω_1 , Ω_2 , the boundary conditions are set on the characteristics $t = \frac{x}{\alpha_i}$, $i = 1, 2$

$$S_1|_{t=\frac{x}{\alpha_1}} = 0, \quad S_2|_{t=\frac{x}{\alpha_2}} = 0 \quad (11)$$

and at the inlet of the porous medium (8).

Below the periodic boundary conditions are considered

$$C_1|_{x=0} = p_1(1 + k_1 \sin(\omega_1 t)), \quad C_2|_{x=0} = p_2(1 + k_2 \sin(\omega_2 t)). \quad (12)$$

3 Exact solution for constant filtration coefficients

Let $\Lambda_i(S) = \lambda_i = \text{const}$. In this case the system (6), (7) splits into two independent systems of the form (1), (2) for each particle type. The exact solution of the problem (6)–(10)

$$C_1 = f_1(t - x / \alpha_1) e^{-\frac{\lambda_1 x}{\alpha_1}}, \quad S_1 = f_1(t - x / \alpha_1) \lambda_1 e^{-\frac{\lambda_1 x}{\alpha_1}} (t - x / \alpha_1), \quad (x, t) \in \Omega_1 \cup \Omega_2;$$

$$C_1 = 0, \quad S_1 = 0, \quad (x, t) \in \Omega_0;$$

$$C_2 = f_2(t - x / \alpha_1) e^{-\frac{\lambda_2 x}{\alpha_2}}, \quad S_2 = f_2(t - x / \alpha_1) \lambda_2 e^{-\frac{\lambda_2 x}{\alpha_2}} (t - x / \alpha_1), \quad (x, t) \in \Omega_2;$$

$$C_2 = 0, \quad S_2 = 0, \quad (x, t) \in \Omega_0 \cup \Omega_1.$$

The exact solution for the concentrations of suspended and retained particles is given by traveling waves with amplitudes $\exp(-\lambda_i x / \alpha_i)$; $i = 1, 2$ that depend on the coordinate x .

4 Asymptotic solution

The asymptotic solution of the problem (6)–(10) is obtained in the form

$$C_i(x, t) = c_i^0(x, t) + c_i^1(x, t) S_{\max} + O(S_{\max}^2), \quad i = 1, 2; \quad (13)$$

$$S_i(x, t) = s_i^1(x, t) S_{\max} + O(S_{\max}^2), \quad i = 1, 2. \quad (14)$$

Substitution of (13), (14) into (6), (7) yields

$$\frac{\partial c_i^0}{\partial t} + \alpha_i \frac{\partial c_i^0}{\partial x} = 0, \quad (15)$$

$$\frac{\partial s_i^1}{\partial t} = \lambda_i^1 (1 - s_1^1 - s_2^1) c_i^0, \quad (16)$$

$$\frac{\partial c_i^1}{\partial t} + \alpha_i \frac{\partial c_i^1}{\partial x} = -\lambda_i^1 (1 - s_1^1 - s_2^1) c_i^0. \quad (17)$$

From (11)–(12) the boundary conditions are determined for equations (15)–(17)

$$s_i^1 \Big|_{t=\frac{x}{\alpha_i}} = 0, \quad s_i^2 \Big|_{t=\frac{x}{\alpha_i}} = 0 \quad (18)$$

$$c_i^0 \Big|_{x=0} = p_i (1 + k_i \sin(\omega_i t)), \quad c_i^1 \Big|_{x=0} = 0, \quad c_i^2 \Big|_{x=0} = 0. \quad (19)$$

The main asymptotic terms are determined successively from the system (15), (19)

$$c_i^0 = \begin{cases} p_i \left(1 + k_i \sin \left(\omega_i \left(t - \frac{x}{\alpha_i} \right) \right) \right), & t > \frac{x}{\alpha_i}; \\ 0, & t < \frac{x}{\alpha_i}; \end{cases} \quad i = 1, 2. \quad (20)$$

Summing the two equations of the system (16) we get

$$\frac{\partial s^1}{\partial t} = (1 - s^1)(\lambda_1 c_1^0 + \lambda_2 c_2^0), \quad s^1 = s_1^1 + s_2^1. \quad (21)$$

The solution of (21) is

$$s^1 = \begin{cases} 1 - e^{-\lambda_1 p_1 \left(t - \frac{x}{\alpha_1} - \frac{k_1}{\omega_1} \left(\cos \left(\omega_1 \left(t - \frac{x}{\alpha_1} \right) - 1 \right) \right) \right) - \lambda_2 p_2 \left(t - \frac{x}{\alpha_2} - \frac{k_2}{\omega_2} \left(\cos \left(\omega_2 \left(t - \frac{x}{\alpha_2} \right) - 1 \right) \right) \right)}, & t > x / \alpha_2; \\ 1 - e^{-\lambda_1 p_1 \left(t - \frac{x}{\alpha_1} - \frac{k_1}{\omega_1} \left(\cos \left(\omega_1 \left(t - \frac{x}{\alpha_1} \right) - 1 \right) \right) \right)}, & x / \alpha_1 < t < x / \alpha_2. \end{cases} \quad (22)$$

5 Numerical calculation

The asymptotic solution is compared with the results of the numerical calculation by the finite difference method for $\alpha_1 = 0.5$; $\alpha_2 = 0.25$; $p_1 = p_2 = 1$; $\Lambda_i(S) = \lambda_i(S_{\max} - S)$, $i = 1, 2$; $\lambda_1 = 1$, $\lambda_2 = 2$; $S_{\max} = 0.1$. The asymptotic terms s_1^1, s_2^1 are determined from (16) numerically by Euler method, c_1^1, c_2^1 are determined from (17) numerically by a counter-current scheme [14].

In Figures 1-6 the graphs of the numerical solution (solid line) and the asymptotic solutions $C_i(x, t) = c_i^0(x, t)$, $i = 1, 2$, $S(x, t) = s^1(x, t)S_{\max}$ are presented (dashed line); the asymptotic solutions $C_i(x, t) = c_i^0(x, t) + c_i^1(x, t)S_{\max}$, $S_i(x, t) = s_i^1(x, t)S_{\max}$, $i = 1, 2$ are also shown (dotted line). At the filter inlet the suspension with variable concentrations $C_i(0, t) = 1 + k_i \sin(\omega_i t)$, $i = 1, 2$ is injected. In Figures 1-3 the parameters $k_1 = -1$, $k_2 = -1$, in Figures 4-6 - $k_1 = -1$, $k_2 = 1$. The 3-D graphs of the solution are given in Figures 3, 6.

Fig. 1. a) Suspended particles concentration $C_1(x, t)|_{x=1}$

b) Suspended particles concentration $C_2(x, t)|_{x=1}$

c) Retained particles concentration $S(x, t)|_{x=1}$

Fig. 2. a) Suspended particles concentration $C_1(x, t)|_{t=1.5}$

b) Suspended particles concentration $C_2(x, t)|_{t=1.5}$

c) Retained particles concentration $S(x, t)|_{t=1.5}$

Fig. 3. a) Suspended particles concentration $C_1(x,t)$

b) Suspended particles concentration $C_2(x,t)$

c) Retained particles concentration $S(x,t)$

Fig. 4. a) Suspended particles concentration $C_1(x,t)|_{x=1}$

b) Suspended particles concentration $C_2(x,t)|_{x=1}$

c) Retained particles concentration $S(x,t)|_{x=1}$

Fig. 5. a) Suspended particles concentration $C_1(x,t)|_{t=1.5}$

b) Suspended particles concentration $C_2(x,t)|_{t=1.5}$

c) Retained particles concentration $S(x,t)|_{t=1.5}$

Fig. 6. a) Suspended particles concentration $C_1(x,t)$

b) Suspended particles concentration $C_2(x,t)$

c) Retained particles concentration $S(x,t)$

6 Conclusion

A model of two particles filtration in the porous medium is considered. Particles of two types are injected with variable concentrations into the porous medium and move with different velocities. Particles freely pass through large pores and get stuck at the inlet of small pores due to the size-exclusion mechanism of particles retention.

For constant filtration coefficients, an exact solution of the problem is obtained in the form of travelling waves of variable amplitude moving in a porous medium with different velocities.

For linear blocking filter coefficients and periodically varying concentrations at the filter inlet, an asymptotic and numerical solution of the problem are obtained.

The obtained analytical and numerical solutions make possible modeling the process of injecting the grout into the porous soil and selecting the optimal parameters for the pumps operation.

References

1. Q. Zhang, P. Li, X. Zhang, S. Li, W. Zhang, Q. Wang S. Li, R. Liu, Q. Zhang, X. Zhang X, *Open Civ. Eng. J.*, **9**, 1 (2015)
2. M. Tsuji, S. Kobayashi, S. Mikake, T. Sato, H. Matsui, *Proc. Eng.*, **191** (2017)
3. L. Faramarzi, A. Rasti, S.M. Abtahi, *Constr. Build. Mat.*, **126** (2016)
4. S.A. Bradford, S. Torkzaban, A. Shapiro, *Langmuir*, **29** (2013)
5. V. Jegatheesan, S. Vigneswaran, *Crit. Rev. Env. Sci. Tec.*, **35**, 6 (2005)
6. C. Tien, B.V. Ramarao, *Granular Filtration of Aerosols and Hydrosols* (Elsevier, Amsterdam, 2007)
7. C.V. Chrysikopoulos, V.I. Syngouna, *J. Environ. Sci. Technol.*, **48** (2014)
8. A.S. Payatakes, R. Rajgopalan, C. Tien, *Can. J. Chem. Eng.*, **52** (1974)
9. G.A. Bartelds, J. Bruining, J. Molenaar, *Transp. Porous Med.*, **26**, 1 (1997)
10. L.I. Kuzmina, Yu.V. Osipov, Y.P. Galaguz, *Int. J. Non-Lin. Mech.*, **93** (2017)
11. L.I. Kuzmina, Yu.V. Osipov, *Proc. Eng.*, **111** (2015)
12. Z. You, Y. Osipov, P. Bedrikovetsky, L. Kuzmina, *Chem. Eng. J.*, **258** (2014)
13. E.A. Vyazmina, P.G. Bedrikovetskii, A.D. Polyanin, *Theor. Found. Chem. Eng.*, **41**, 5 (2007)
14. L.I. Kuzmina, Yu.V. Osipov, Y.P. Galaguz, *Int. J. Comp. Civ. Struct. Eng.*, **13**, 3 (2017)