Assessment of shear capacity. Part I: Comparison of shear models used for RC beams design

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Abstract. This paper briefly describes commonly used methods of determination of shear strength of reinforced concrete beams. Classical method, Eurocode 2 approach and most complex approach proposed by P. D. Zararis in a series of papers are depicted and compared. Experimental tests of RC beams shown in the subsequent paper (Part II) are confronted with the model predictions.

1 Introduction

Theoretical models used in structural design should be able to estimate element strength in the way, which is both simple and general. They also have to find correspondence with the real structure behaviour under variety of load cases. At the same time they need to preserve sufficient safety factor, not allowing limit capacity states being achieved.

Despite of numerous theoretical research and experimental results, the issue of shear failure in beams and other reinforced concrete (RC) elements remains unresolved completely. The problem is complicated due to multiaxial and inhomogeneous stress distribution in RC elements. Therefore, analytical formulas based on simple models, which could predict element capacity precisely enough, have not yet been formulated.

In this paper three models of shear capacity determination are presented. Then examples of model predictions are given.

2 Classical method

Classical method of shear capacity evaluation in RC beams is based on E. Mörsch works [1]. Author assumed there that RC element works in the cracked phase no. II – concrete section carries compressive stress only. The assumption implicates, that principal stress trajectories in the shear zone can be considered as straight lines, inclined by the angle of 45° to the horizontal axis. Pattern of cracking observed in real elements, confirms the thesis. In such a case, the shear stress below the neutral axis takes value:

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$$\tau_{\max} = \tau_0 = \frac{V}{bz}, \quad (1)$$

where V is the shear force, b is the width of the cross-section, z is the lever arm of internal forces, for a member with constant depth, corresponding to the bending moment in the element.

Inclined cracks open when the principal tensile stress in the shear zone exceeds concrete tensile force. Cracks divide the beam to compressive concrete struts inclined by the angle of 45° to the horizontal axis. Stirrups can be considered as truss web members which leads to the truss analogy near the support. Bottom layer of the longitudinal reinforcement subjected to tensile forces, acts as bottom chord, when compressed top chord is represented by the concrete.

2.1 Determination of stirrups ultimate force

In consequence of Mörsch [1] truss analogy, forces in stirrups can be determined by Ritter method from the equilibrium of forces acting on a part of beam separated by the inclined crack, shown in Figure 1.

Fig. 1. a) Support zone with inclined crack b) Forces acting on a part of beam divided by inclined crack.

From vertical force equilibrium, forces in stirrups can be obtained:

$$R - \sum_{i=1}^n F_{st,i} = 0 \quad \text{hence} \quad F_{st} = \frac{R}{n}. \quad (2)$$

Taking $R = V_{Ed}$ and $n = \frac{z}{s}$ in (2) leads to:

$$F_{st} = \frac{V_{Ed}s}{z}, \quad (3)$$

where V_{Ed} is the design value of the applied shear force, s is the spacing of the stirrups, n is the number of the stirrups.

Stirrup ultimate force is expressed as follows:

$$F_{st,max} = f_{ywd} A_{st}, \quad (4)$$

where f_{ywd} is the design yield strength of the shear reinforcement, A_{st} is the section area of the shear reinforcement.

Therefore, the ultimate stirrup force condition is given:

$$F_{st} \leq F_{st,max}. \quad (5)$$

Taking into account (3) and (4), the final condition can be written as follows:

$$V_{Ed} \leq f_{ywd} A_{st} \frac{z}{s} . \tag{6}$$

2.2 Determination of ultimate force in compressive concrete struts

Forces in concrete struts can be obtained based on Figure 2. The width of concrete inclined strut takes value of $z\sqrt{2}$ and its horizontal projection value equals z .

Fig. 2. Stress value in compressive concrete chord.

Maximum compressive force can be obtained from the concrete compressive strength:

$$F_{c,u} = f_{cd}bz\sqrt{2} \text{ then } F_{c,max} = F_{c,u} \sin 45^{\circ} = f_{cd}bz\sqrt{2} \frac{1}{\sqrt{2}} = bz f_{cd} , \tag{7}$$

where f_{cd} is the design value of the concrete compressive strength.

Concrete struts ultimate capacity condition takes form:

$$V_{Ed} \leq bz f_{cd} . \tag{8}$$

2.3 Shear strength of beams without shear reinforcement

Shear strength condition near the support zone of beams without shear reinforcement [2] is fulfilled, as inequality (9) holds true:

$$V_{Ed} \leq 0,75bdf_{ctd} , \tag{9}$$

where f_{ctd} is the design value of the concrete tensile strength.

3 Shear strength of RC beams according to Eurocode 2

Eurocode approach to determination of shear strength in RC beams evolved from methods based on the truss analogy [3-5]. Internal force state is identified with the truss shown on Figure 3, where the top chord is represented by compressed concrete and the bottom chord by longitudinal reinforcement subjected to tension [4]. Between the horizontal forces F_{cd} and F_{ctd} there is a shear zone containing compressed concrete struts divided by cracks, inclined to the horizontal axis by angle of θ and the shear reinforcement inclined by angle of α .

Fig. 3. Truss model according to Eurocode 2 [4].

Shear strength is defined by three conditions, determining ultimate shear force: due to strength of beam without shear reinforcement, due to stirrups strength and due to strength of concrete struts subjected to compressive stress:

$$V_{Ed} \leq V_{Rd,c} = \max \left\{ [c_{Rd,c} k (100 \rho_L f_{ck})^{1/3} + k_1 \sigma_{cp}] b_w d, (v_{\min} + k_1 \sigma_{cp}) b_w d \right\}, \quad (10)$$

$$V_{Ed} \leq V_{Rd,s} = \frac{A_{sw}}{s} z f_{ywd} \cot \theta, \quad (11)$$

$$V_{Ed} \leq V_{Rd,max} = \frac{\alpha_{cw} b_w z v_1 f_{cd}}{\cot \theta + \tan \theta}, \quad (12)$$

where V_{Ed} is the design value of the applied shear force, $V_{Rd,c}$ is the design shear strength of the member without shear reinforcement, $V_{Rd,s}$ is the design value of the shear force which can be sustained by the yielding shear reinforcement, $V_{Rd,max}$ is the design value of the maximum shear force which can be sustained by the member, limited by crushing of the compression struts, $c_{Rd,c}$, k , k_1 are the factors calculated according to EC2, ρ_L is the reinforcement ratio for the longitudinal reinforcement which extends $\geq (l_{bd} + d)$, σ_{cp} is the concrete compressive stress at the centroidal axis due to axial loading and (or) prestressing, A_{sw} is the area of the shear reinforcement, α_{cw} is a coefficient taking account of the state of the stress in the compression chord.

Values of forces V_{Rd} obtained from (11) and (12) are variable, depending from the inclination angle of concrete strut to the horizontal axis θ , which values are restricted by $\cot \theta \in <1.0 \div 2.5>$ [4].

4 Shear strength according to Zararis

Less known approach to the shear capacity assessment is the one proposed by P. D. Zararis in works [6-9]. He described a model of RC beam simply supported, loaded by two-point forces. Theory is based on analysis of inclined crack propagation in complex stress state, including shear force and bending moment effects and depicting two branches of crack. The first crack opens below the neutral axis after formation of flexural cracking and develops in the direction inclined to vertical axis (Figure 4). The second branch initiates from the tip of the initial branch and develops to the force location point. Both branches form the critical crack leading to beam failure due to shear forces.

Fig. 4. Two branches of critical crack.

From stress state analysis in the shear zone just after the initial crack forms (phase II), angle between vertical axis and the first branch of critical crack φ can be obtained, together with compressive zone height of x and distance from the support to the initial crack opening

point y . Full algorithm of calculations can be found in works [6-9], but a brief description is given below.

Fig. 5. Formation of initial crack.

Values presented in Figure 5, x and φ are analytically derived with the following assumptions: 1) direction of the longitudinal reinforcement strain is perpendicular to crack width direction, 2) reinforcement bars deform according to linear-elastic phase, 3) normal stresses distribution in concrete compressive zone is parabolic, 4) concrete top fibres reach ultimate stress value, corresponding to the maximum stress, according to EC2 [4] $\varepsilon_{c0} = \varepsilon_{c2} = 0.002$, 5) Poisson ratio and Young modulus for steel are taken as: $\nu = 0.3$, $E_s = 200$ GPa.

4.1 Shear strength of beam without transversal reinforcement

Height of the inclined crack is derived from second order equation (13), describing force state at a portion of beam divided by inclined crack [7], shown in Figure 6.

Fig. 6. Force state diagram at inclined crack.

$$\left(\frac{x}{d}\right)^2 + 600 \frac{\rho x}{f_{cd} d} - 600 \frac{\rho}{f_{cd}} = 0. \tag{13}$$

Angle of the first branch of inclined crack can be found from equilibrium of a part of the beam shown in Fig. 7a, taking value of $\lambda = 0.72$ [8]. Distance from the support to the critical crack x is derived from geometrical correspondence, shown in Figure 7b.

Fig. 7. a) Line of compressive action in shear zone. b) Distance from support to critical crack [8].

$$\tan^2 \varphi - \frac{a}{d-x} \tan \varphi + \frac{2.5d-x}{d-x} = 0, \tag{14}$$

Assuming align to the second branch, where the only compressive force acts [8], as shown in Figure 8, analogy to the compression of a circular disk loaded by a group of self-

equilibrated forces is used (Figure 9a). Stress distribution along the disk diameter is shown in Figure 9b.

Fig. 8. Force state diagram at second branch of critical crack.

Fig. 9. a) Analogy of self-equilibrated forces acting on circular disk. b) Normal stresses distribution along disk diameter. [8]

Normal stress at point D is always tensile, the value can be derived as follows [8]:

$$\sigma_D = \frac{2F}{\pi b d_{cr}} = \frac{2 \frac{P}{\sin \vartheta}}{\pi b l_{cr}} = \frac{2P}{\pi b l_{cr} \sin \vartheta}, \quad (15)$$

where ϑ is an angle between the second branch of critical crack and horizontal direction.

When the value of tensile stress at point D exceeds tensile strength of concrete f_{ctd} , the second branch starts to form. As diameter length becomes smaller, crack propagates towards the load location, leading to failure due to shear action. Ultimate transverse force, which substitutes shear strength of beam without shear reinforcement, can be written as follows:

$$V_{Rd,c} = b x f_{ctd}. \quad (16)$$

Exceeding ultimate force value implicates immediate beam failure caused by increase of angle φ , which increases force in transverse reinforcement, and causes horizontal cleavage of concrete cover along bottom level of the longitudinal reinforcement and loss of its transversal strength. As a result, the whole shear force is taken by concrete above the second branch of critical crack. Then, shear surface forms from the load location to the support.

Experimental observations [7] showed, that beam shear strength decreases when distance from the support to the load point increases, size effect correction to (16) is proposed as:

$$V_{Rd,c} = (1.2 - 0.2a) b x f_{ctd}, \quad (17)$$

where a is the distance from the load location to the support and $1.2 - 0.2a \geq 0.65$.

4.2 Ultimate shear force in beam with transversal reinforcement

Stirrups contribution to shear strength is taken just after the second branch of inclined crack forms as shown on Figure 10, where A_{st} the area of shear reinforcement, s – stirrups spacing, $f_{yd,st}$ – is the design yield strength of shear reinforcement, l_t – length over which shear force in longitudinal reinforcement is lost, ΔV_s – change of shear force in longitudinal bars due to splitting of concrete cover, can be described as follows:

$$\Delta V_{Rd} = 0.5 \rho_{st} f_{yd,st} b d , \tag{18}$$

where ρ_{st} is transversal reinforcement ratio.

Fig. 10. Shear forces in reinforcement – case with stirrups [9].

To get force in stirrups V_{Rd} , it is assumed that when the critical crack opens, normal stresses in stirrups increase linearly along the crack surface. Maximum value of steel yield strength $f_{yd,st}$ is reached at the initiation point of the crack. For simplicity length of crack can be approximated by $0.5a$, what implicates formula:

$$V_{Rd} = 0.5 A_{st} f_{yd,st} \frac{0,5a}{s} = 0.25 \rho_{st} f_{yd,st} a b , \tag{19}$$

and leads to the ultimate shear strength of beam with transversal reinforcement:

$$V_{Rd} = \left[(1.2 - 0.2a) \frac{x}{d} f_{ctd} + \left(0.25 \frac{a}{d} + 0.5 \right) \rho_{st} f_{yd,st} \right] b d . \tag{20}$$

5 Examples of models prediction

To illustrate and compare described methods of RC beams shear strength determination, calculations have been made for the chosen example with the following data: cross-section dimensions: $b = 80$ mm; $h = 140$ mm; reinforcement bar diameters: main $\phi = 8$ mm; stirrups: $\phi_{st} = 6$ mm; effective section height: $d = 120$ mm; shear span length: $a = 300$ mm; partial factors for concrete and steel: $\gamma_c = 1.4$; $\gamma_s = 1.15$. Material parameters: concrete $f_{ck} = 30$ MPa, $f_{ctk} = 2,9$ MPa, steel B500SP $f_{yk} = 500$ MPa. Stirrup spacing: $s = 90$ mm, angle of θ has been taken to satisfy condition: $\cot \theta \in (1,0 \div 2,5)$.

Obtained results are presented in Table 1.

Table 1. Shear capacity of RC beam due to sub-elements strength.

	Classical method	Eurocode 2	Zararis method
Strength of concrete without shear reinforcement $V_{Rd,c}$	14.91 kN	7.78	9.39 kN
Strength due to crushing compressive concrete struts $V_{Rd,max}$	185.14 kN	33.71kN for $\theta=21.8^\circ$ 48.88 kN for $\theta= 45^\circ$	-
Yield strength of transversal reinforcement $V_{Rd,s}$	27.32 kN	29.50 kN for $\theta= 45^\circ$ 73.76 kN for $\theta=21.8^\circ$	36.88 kN

Results shown in Table 1 implicate, that the most rigorous condition on the shear capacity for a beam without transversal reinforcement is imposed by EC2 [4]. Although stirrup's contribution highly increases beam strength according to EC2, with precise selection of concrete strut inclination angle – θ , much more than other methods. Moreover, high range of possible selection in θ (from 21.8° to 45°), can change shear capacity significantly.

The highest value of concrete strut strength can be obtained from the classical method, while the Zararis method is not applicable to the case.

6 Conclusion

Described shear strength models are of different level of complexity, what results in incompatibilities in shear capacity predictions according to each of them.

The simplest model for shear verification, used in classical method cannot describe failure due to strength of concrete without transversal reinforcement, while with contribution of stirrups, simplifies their behaviour to one-dimensional problem, which underestimates predicted value of the shear strength.

Model proposed by EC2 [4], due to its common use by structural designers should lead to safe results in each design situation. It includes the highest number of failure mechanisms according to strength of each beam component. However, to reach that wide range of application, dimensioning of each element is simplified and a lot of factors is given without analytical explanation.

The most complex model, created by Zararis [6-9], enables better use of material in low or medium reinforced elements. On the other hand, some formulas come from simplifications and experimental results, moreover, only selected failure mechanisms are included and described.

Despite of numerous theoretical research and experimental results, no analytical formulas, which could predict element strength precisely enough, have been formulated. None of the common-used methods includes all design situations. Most of models are based on simplifications and experimentally based factors, even new and complex ones, like the described Zararis approach.

As a result of undertaken research and carried out calculations, authors assume, that problem of shear in RC elements remains unresolved completely. It is believed that dimensioning methods due to shear can be improved or replaced by new models, simple in use and universal.

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