Finite elements with continuous stress fields in calculation of folded prismatic thin-walled rods and shells

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Abstract. The article deals with methods of creating a rectangular wall-beam finite element with eight degrees of freedom per node and continuous stress fields along the boundaries. This effect is achieved by specifying displacement fields in the plane of the element in forms similar to those in finite elements of Bogner, Fox, and Schmitt plate. The article provides algebraic expressions for displacement forms; methods of forming reaction and stress matrices are also considered. Test calculations carried out with the help of "Computational mechanics" FEM complex have proved high efficiency of the finite element analysis performed. A rectangular shell finite element with twelve degrees of freedom per node was developed as a combination of membrane finite element and Bogner, Fox and Schmitt plate element.

1 Introduction

Finite element method (FEM) is often used for modeling various types of complex structures. But in calculations of flat and folded thin-walled structures such as open section and box section bridge beams with orthotropic carriageway plates, or frames and trusses composed of thin-walled rods with cross section deformation and stress from constrained torsion, finite element method has a number of disadvantages that reduce the reliability of the analysis. Among them are gaps in stress fields along the element's boundaries and at nodal points. Due to considerable length of such structures, FEM models formed with the help of conventional finite elements like plates and shells (for example, in [1]) require a very large number of elements to carry out precise calculations. As a result, modeling process becomes very complicated. Square-shaped plates that perfectly suit ordinary rectangular finite elements tend to produce less accurate calculation results when the ratio of length and width of rectangular elements increases.

To overcome these drawbacks the authors have introduced a rectangular wall-beam finite element with eight degrees of freedom per node (see [2]), as part of "Computational mechanics" finite element software package created at the CAD Department of Moscow State University of Railway Engineering (MIIT) under the

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guidance of Prof. N.N. Shaposhnikov [3] and later developed into "Katran" [4] FEM package. Software testing and comparison of the results with those given in R. Gallagher's book [5] have shown that the rectangular finite element has considerable advantages and surpasses conventional wall-beam elements in many ways.

2 Basic assumptions and formulas

In order to create a rectangular membrane finite element with eight degrees of freedom per node we set the components of displacement fields in the plane of the plate and specify complete cubic polynomials of displacement interpolation functions introduced by Bogner, Fox and Schmitt [6].

Figure 1 represents shape function graphs of a rectangular beam-wall finite element described in [2]. The shape functions displayed as projections on side edges of the rectangle are the product of functions n_{ix} and n_{iy}

The cubic polynomials appear as components of specified displacement shapes along the axes of the element's local coordinate system for node *i*: U_i along *X* axis and V_i along *Y* axis. The value of shape functions in rectangle vertices and their first derivatives is, respectively, 1 or 0. Shape function graphs of other nodes are similar to the above mentioned. Their expressions are shown in Table 1.

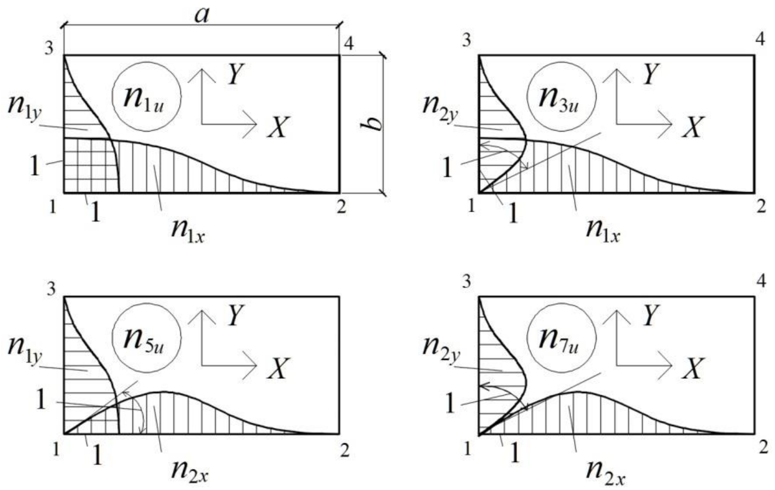

Fig. 1. Graphs of shape functions n_{1u}, n_{3u}, n_{5u}, n_{7u} of possible U_1 displacements along *X* axis for four of the eight degrees of freedom per node, specified for node 1 of the rectangular membrane finite element.

Graphs of shape functions n_{2v}, n_{4v}, n_{6v}, n_{8v} of possible V_i displacements along *Y* axis for other four degrees of freedom in node 1 are similar to those in figure 1, with the exception of indexes in function names. The local coordinate system center of the rectangular element coincides with the intersection point of its symmetry axes.

Displacement fields that satisfy the requirements of continuity of stress fields in neighboring finite elements, as well as continuity and completeness of displacements, are described by the following formulas where C_i are constants:

$$U = C_1\, n_{1u} + C_3\, n_{3u} + \ldots + C_{31}\, n_{31u} \tag{1}$$

$$V = C_2\, n_{2v} + C_4\, n_{4v} + \ldots + C_{31}\, n_{32u} \tag{2}$$

Table 1. Shape functions that describe displacement fields of a membrane rectangular finite element with eight degrees of freedom per node, for Nodes 1 - 4.

	Direction u_i	Direction v_i	Direction u_i	Direction v_i
Node 1	$n_{1u} = n_{1x}\, n_{1y}$	$n_{2v} = n_{1x}\, n_{1y}$	$n_{3u} = n_{1x}\, n_{2y}$	$n_{4v} = n_{2x}\, n_{1y}$
	$n_{5u} = n_{2x}\, n_{1y}$	$n_{6v} = n_{1x}\, n_{2y}$	$n_{7u} = n_{2x}\, n_{2y}$	$n_{8v} = n_{2x}\, n_{2y}$
Node 2	$n_{9u} = n_{3x}\, n_{1y}$	$n_{10v} = n_{3x}\, n_{1y}$	$n_{11u} = n_{3x}\, n_{2y}$	$n_{12v} = n_{4x}\, n_{1y}$
	$n_{13u} = n_{4x}\, n_{1y}$	$n_{14v} = n_{3x}\, n_{2y}$	$n_{15u} = n_{4x}\, n_{2y}$	$n_{16v} = n_{4x}\, n_{2y}$
Node 3	$n_{17u} = n_{1x}\, n_{3y}$	$n_{18v} = n_{1x}\, n_{3y}$	$n_{19u} = n_{1x}\, n_{4y}$	$n_{20v} = n_{2x}\, n_{3y}$
	$n_{21u} = n_{2x}\, n_{3y}$	$n_{22v} = n_{1x}\, n_{4y}$	$n_{23u} = n_{2x}\, n_{4y}$	$n_{24v} = n_{2x}\, n_{4y}$
Node 4	$n_{25u} = n_{3x}\, n_{3y}$	$n_{26v} = n_{3x}\, n_{3y}$	$n_{27u} = n_{3x}\, n_{4y}$	$n_{28v} = n_{4x}\, n_{3y}$
	$n_{29u} = n_{4x}\, n_{3y}$	$n_{30v} = n_{3x}\, n_{4y}$	$n_{31u} = n_{4x}\, n_{4y}$	$n_{32v} = n_{4x}\, n_{4y}$

Where:

$$n_{1x} = 2\xi^3 - {}^{3}\!/_{2}\,\xi + {}^{1}\!/_{2} \tag{3}$$

$$n_{2x} = a\xi^3 - {}^{a}\!/_{2}\,\xi^2 - {}^{a}\!/_{4}\,\xi + {}^{a}\!/_{8} \tag{4}$$

$$n_{3x} = -2\xi^3 + {}^{3}\!/_{2}\,\xi + {}^{1}\!/_{2} \tag{5}$$

$$n_{4x} = a\xi^3 + {}^{a}\!/_{2}\,\xi^2 - {}^{a}\!/_{4}\,\xi - {}^{a}\!/_{8} \tag{6}$$

$$\xi = {}^{x}\!/_{a}; \quad \eta = {}^{y}\!/_{b} \tag{7}$$

n_{iy} functions are basically the same, though ξ is changed to η and a is changed to b. Note that the functions satisfy the deformation compatibility equation. We can prove it by substituting them into the expression:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{8}$$

where:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{bmatrix} = \mathbf{B\,Z}, \tag{9}$$

Z – vector of constants,

$$\mathbf{B} = \begin{bmatrix} \partial n_{1u} / \partial x & 0 & \dots & 0 \\ 0 & \partial n_{2v} / \partial y & \dots & \partial n_{32v} / \partial y \\ \partial n_{1u} / \partial y & \partial n_{2v} / \partial x & \dots & \partial n_{32v} / \partial x \end{bmatrix} \tag{10}$$

Using formulas given in [3], we obtain an expression for the finite element's reaction matrix:

$$\mathbf{r} = \delta \iint \mathbf{B_T\, D\, B}\, dx\, dy, \tag{11}$$

where:

D stands for Hooke's law relationship in matrix form, $\mathbf{B_T}$ for the transpose of matrix **B**, δ – for plate thickness.

The integration of expression (11) presents certain difficulties due to the large size of matrix **B**. To simplify the integration of the expressions in Table 1 we present matrix **B** (3×32) as a product:

$$\mathbf{B} = \mathbf{B_1}\,\mathbf{L}, \tag{12}$$

where:
matrix $\mathbf{B_1}$ includes degrees of ξ, η, its size is (3 × 39);
matrix **L** contains coefficients of ξ, η, it obtained as a result of differentiation of expression (10) and its size is (39 × 32).

After multiplying $\mathbf{B_1}$ by **L**, we obtain expressions for the elements of matrix **B** using the formula (10).

After formula (12) is replaced with expression (11), we acquire matrix **S**:

$$\mathbf{S} = \iint \mathbf{B_{1T}}\,\mathbf{D}\,\mathbf{B_1}\, d\xi\, d\eta \tag{13}$$

The size of matrix **S** is (39 × 39); since integrals of odd degrees of ξ, η equal to zero, it contains a number of zero elements, given that a local coordinate system of the finite element is used (see figure 1).

As a result of these transformations, we obtain a matrix expression for formulas defining r_{ij}, elements of reaction matrix **r** of the finite element**:**

$$\mathbf{r} = a\, b\, \delta\, \mathbf{L_T}\,\mathbf{S}\,\mathbf{L} \tag{14}$$

There exists an alternative way of forming a reaction matrix of a finite element with eight degrees of freedom per node which does not require differentiation of basic functions and subsequent integration. If we substitute basic functions and elements d_x and G of Hooke's law matrix **D** into the integral expression (11), we obtain a formula for the element r_{11} of the finite element reaction matrix:

$$r_{11} = \delta \iint \left(d_x \frac{\partial n_{1x}}{\partial x} \frac{\partial n_{1x}}{\partial x}\, n_{1y}\, n_{1y} + G \frac{\partial n_{1y}}{\partial y} \frac{\partial n_{1y}}{\partial y}\, n_{1x}\, n_{1x}\right) dx\, dy \tag{15}$$

To calculate the value of integral (15) we use a table whose components are available in a number of sources, for example, in [6]. The structure of the table is shown in figure 2.

Formulas similar to the expression (15) can also be obtained for other components of reaction matrix r_{ij}. To calculate these elements we need to fill in $\mathbf{S_n}$ matrix (see figure 2) and then the next five matrices whose size is (3 × 32):
$\mathbf{D_{dT}}$ is filled with elements of Hooke's law matrix in accordance with expression (15) by columns headed with numbers of degrees of freedom of the finite element, then transposed;
$\mathbf{B_{xdT}}$ - filled by rows with components of table (see figure 2) for function n_{ix} and its first derivative n_{ix}' and by degrees of freedom of the finite element; transposed;
$\mathbf{B_{ydT}}$ - the same for functions n_{iy} and n_{iy}'; transposed;
$\mathbf{B_x}$ - for functions n_{ix} and n_{ix}';
$\mathbf{B_y}$ - for functions n_{iy} and n_{iy}'.

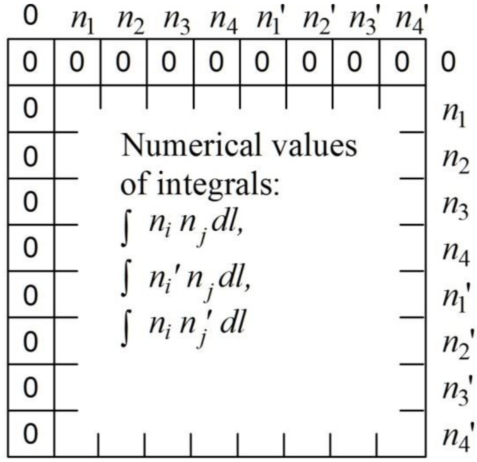

Fig. 2. Matrix $\mathbf{S_n}$, table of values of integrals from products of basis functions and their derivatives.

Thus, the procedure for calculating values of elements of the reactions matrix of the finite element is reduced to operations with matrices: $\mathbf{S_n}$, $\mathbf{D_d}$, $\mathbf{B_{xd}}$, $\mathbf{B_{yd}}$, $\mathbf{B_x}$, $\mathbf{B_y}$.

Stress values at the points of the finite element are calculated by the formula:

$$[\sigma_x \;\; \sigma_y \;\; \tau_{xy}]_\mathbf{T} = \mathbf{D\, B\, Z} = \mathbf{D\, B_1\, L\, Z} \qquad (16)$$

Point coordinates for calculating stress values by the formula (16) can be defined by dividing the finite element into a set of rectangular subelements according to topology matrix $\mathbf{T_e}$ of nested subelements and the matrix $\mathbf{C_e}$ of their vertices' coordinates. The number of nested rectangles is determined by number of rows n and columns m. After setting the values for n and m, the components of matrices $\mathbf{T_e}$ and $\mathbf{C_e}$ are calculated by parametric formulas using an appropriate software solution.

In "Computational mechanics" FEM software package source data and final results are generated as text files, but the image of the calculation model and graphical results can be obtained by AutoCAD using the techniques and solutions given in [7]. It is possible to create visualizations of stress fields for the wall-beam finite element in their continuity without additional approximating procedures.

3 Conclusion

The research has shown that a wall-beam finite element with eight degrees of freedom per node has the following advantages:

- it produces stress fields without jump discontinuities at the boundaries with other elements and nodes;
- allows the use of elements with size ratio (a / b) about 100 without losing the precision of the analysis;
- gives more accurate results with fewer total degrees of freedom;
- helps to calculate reliable stress values on the outer edges of the finite element model;
- introducing a rectangular finite element of Bogner, Fox and Schmitt plate as bending component, we can develop a shell finite element with twelve degrees of freedom per node and continuous displacement fields.

References

1. V.S. Karpilovskyy, E.Z. Kryksunov, A.A. Maliarenko, A.V. Perelmuter, M.A. Perelmuter, S.Y. Fialko, *SCAD Office. V.21. System SCAD++* (Publishing House SCAD SOFT, Moscow, 2015)
2. S.N. Nazarenko, VINITI Dep. 3052-V93 (1993)
3. A.V. Darkov, N.N. Shaposhnikov, *Structural mechanics* (LAN, SPb., 2010)
4. N.N. Shaposhnikov, I.V. Nesterov, Vest. MGSU, **3**, 57 (2012)
5. R. Gallagher, *Finite element analysis*. *Fundamentals* (Mir, Moscow, 1984)
6. A.S. Gorodetski, I.V. Zavaritskii, A.I. Lantukh-Lyashchenko, A.O. Rasskazov, *Automation of calculations of transport constructions* (Transport, Moscow , 1989)
7. S.N. Nazarenko, M.A. Gurkova, *Automation of creation of drawings by AutoCAD system* (MIIT, Moscow, 2009)