Calculation of orthotropic plates for creep taking into account shear deformations

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Abstract. A system of differential equations is obtained for calculating the creep of orthotropic plates taking into account the deformations of the transverse shear. The basic hypothesis is a parabolic change in tangential stresses over the thickness of the plate. An example of the calculation is given for a GRP plate hinged on the contour under the action of a uniformly distributed load.

1 Introduction

The classical theory of plates, constructed on the hypothesis of the straight normal, describes well the stress-strain state of only those thin plates in which the deformations of transverse shear γ_{xz} and γ_{yz} are significantly smaller than the rotation angles $\frac{\partial w}{\partial x}$ and

$\frac{\partial w}{\partial y}$. There is a large number of papers devoted to creep calculation of plates based on the Kirchhoff-Love theory, including [1-2]. In isotropic plates, the shear angles are extreme near the reference contour, points and lines of application of concentrated and distributed along the lines forces, as well as near the notches [3]. The classical theory gives the greatest errors in stress values in these zones.

However, if the thin plate is anisotropic and the transverse shear moduli are significantly smaller than the longitudinal elastic moduli in directions parallel to the median plane, then the shear angles γ_{xz} and γ_{yz} are comparable with the rotation angles far from the indicated regions, i.e. in the inner region of the plate. Because of this, the classical theory leads to noticeable errors at all points of the plate.

In the literature there is a large number of different versions of refined theories that differ not only in the hypotheses and factors used, but also in approaches to their construction.

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2 Methods

According to [3,4], the least errors are given by theories based on the hypothesis of a parabolic change in the stresses τ_{xz} and τ_{yz} over the plate thickness. One of these theories, which also takes into account the normal stresses σ_z in addition to the parabolic distribution of the tangential stresses, is presented in the paper of S.A. Ambartsumian [4]. In this paper, we generalize theory of S.A. Ambartsumian for viscoelastic material.

Following [4], the tangential stress distribution is taken as:

$$\tau_{xz} = \frac{\partial \varphi}{\partial x} \left(1 - 4z^2 / h^2\right); \quad \tau_{yz} = \frac{\partial \varphi}{\partial y} \left(1 - 4z^2 / h^2\right), \quad (1)$$

where $\varphi = \varphi(x, y)$ is the desired shift function with the force dimension.

The relationship between strains and stresses will be written as:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}}{E_y} \sigma_y - \frac{\nu_{zx}}{E_z} \sigma_z + \varepsilon_x^*, \quad \varepsilon_y = -\frac{\nu_{xy}}{E_x} \sigma_x + \frac{\sigma_y}{E_y} - \frac{\nu_{zy}}{E_z} \sigma_z + \varepsilon_y^*, \quad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} + \gamma_{xy}^*. \quad (2)$$

The index * here and below corresponds to the creep strains.

We express from (2) the stresses through deformations:

$$\begin{aligned} \sigma_x &= \frac{E_x}{1 - \nu_{xy}\nu_{yx}} \left[\varepsilon_x + \nu_{yx}\varepsilon_y - \left(\varepsilon_x^* + \nu_{yx}\varepsilon_y^* \right) \right] + A_x \sigma_z; \\ \sigma_y &= \frac{E_y}{1 - \nu_{xy}\nu_{yx}} \left[\varepsilon_y + \nu_{xy}\varepsilon_x - \left(\varepsilon_y^* + \nu_{xy}\varepsilon_x^* \right) \right] + A_y \sigma_z; \quad \tau_{xy} = G_{xy} (\gamma_{xy} - \gamma_{xy}^*), \end{aligned} \quad (3)$$

$$\text{where } A_x = \frac{\nu_{xy}\nu_{yz} + \nu_{xz}}{1 - \nu_{yx}\nu_{xy}}; \quad A_y = \frac{\nu_{yz} + \nu_{xz}\nu_{yx}}{1 - \nu_{yx}\nu_{xy}}.$$

The stresses τ_{xz} , τ_{yz} and σ_z are related by the equilibrium equation:

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \quad (4)$$

Substituting (1) into (4), we obtain:

$$\frac{\partial \sigma_z}{\partial z} = -\nabla^2 \varphi \left(1 - 4z^2 / h^2\right). \quad (5)$$

The integration of equation (5) yields:

$$\sigma_z = -\nabla^2 \varphi \Phi(z) + C(x, y), \quad (6)$$

$$\text{where } \Phi(z) = z \left(1 - 4z^2 / (3h^2)\right).$$

The function $C(x, y)$ is determined from the boundary conditions on the plate surfaces:

$$\sigma_z(x, y, h/2) = 0; \quad \sigma_z(x, y, -h/2) = -q(x, y), \quad (7)$$

which leads to the formula:

$$\sigma_z(x, y, z) = -\frac{1}{2} q(x, y) - \Phi(z) \nabla^2 \varphi(x, y) \quad (8)$$

and to the integral equation of equilibrium of the plate element $hdx dy$ with respect to the z axis:

$$\nabla^2 \varphi = -\frac{3}{2} \frac{q(x, y)}{h}. \quad (9)$$

Equation (9) allows expressing the stresses σ_z explicitly through the load:

$$\sigma_z = -\frac{q}{2} \left(1 - \frac{3z}{h} + \frac{4z^3}{h^3} \right). \quad (10)$$

The Cauchy relations for shear strains have the form:

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}. \quad (11)$$

From the physical point of view, the deformations γ_{xz} and γ_{yz} represent the sum of elastic deformations and creep strains:

$$\gamma_{xz} = \frac{\tau_{xz}}{G_{xz}} + \gamma_{xz}^*; \quad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} + \gamma_{yz}^*. \quad (12)$$

Substituting (1) into (12) and equating (12) to (11) then, we obtain:

$$\frac{\partial u}{\partial z} = \frac{1}{G_{xz}} \frac{\partial \varphi}{\partial x} \left(1 - \frac{4z^2}{h^2} \right) - \frac{\partial w}{\partial x} + \gamma_{xz}^*. \quad (13)$$

Integration of the equality (13) with respect to z gives:

$$u = \frac{\Phi(z)}{G_{xz}} \frac{\partial \varphi}{\partial x} + \int_0^z \gamma_{xz}^* dz - z \frac{\partial w}{\partial x} + f_1(x, y). \quad (14)$$

The median plane of the plate is assumed to be non-deformable, i.e. for $z = 0$ $u = 0$, from which $f_1(x, y) = 0$. Similarly, for displacements v , we can write:

$$v = \frac{\Phi(z)}{G_{yz}} \frac{\partial \varphi}{\partial y} - z \frac{\partial w}{\partial y} + \int_0^z \gamma_{yz}^* dz. \quad (15)$$

Deformations ε_x , ε_y and γ_{xy} are defined as follows:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} + \frac{\Phi(z)}{G_{xz}} \frac{\partial^2 \varphi}{\partial x^2} + \Gamma_x; \quad \varepsilon_y = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} + \frac{\Phi(z)}{G_{yz}} \frac{\partial^2 \varphi}{\partial y^2} + \Gamma_y; \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial x \partial y} \Phi(z) \left(\frac{1}{G_{xz}} + \frac{1}{G_{yz}} \right) + \Gamma_{xy}, \end{aligned} \quad (16)$$

where $\Gamma_x = \frac{\partial}{\partial x} \int_0^z \gamma_{xz}^* dz$; $\Gamma_y = \frac{\partial}{\partial y} \int_0^z \gamma_{yz}^* dz$; $\Gamma_{xy} = \frac{\partial}{\partial x} \int_0^z \gamma_{yx}^* dz + \frac{\partial}{\partial y} \int_0^z \gamma_{xz}^* dz$.

Substituting (16) into (3), we obtain:

$$\begin{aligned} \sigma_x &= -\frac{E_x}{1 - \nu_{xy} \nu_{yx}} \left[z \left(\frac{\partial^2 w}{\partial x^2} + \nu_{yx} \frac{\partial^2 w}{\partial y^2} \right) + \varepsilon_x^* + \nu_{yx} \varepsilon_y^* - \left(\Gamma_x + \nu_{yx} \Gamma_y \right) \right] + \\ &\quad + \frac{E_x \Phi}{1 - \nu_{xy} \nu_{yx}} \left(\frac{1}{G_{xz}} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\nu_{yx}}{G_{yz}} \frac{\partial^2 \varphi}{\partial y^2} \right) + A_x \sigma_z; \\ \sigma_y &= -\frac{E_y}{1 - \nu_{xy} \nu_{yx}} \left[z \left(\frac{\partial^2 w}{\partial y^2} + \nu_{xy} \frac{\partial^2 w}{\partial x^2} \right) + \varepsilon_y^* + \nu_{xy} \varepsilon_x^* - \left(\Gamma_y + \nu_{xy} \Gamma_x \right) \right] + \\ &\quad + \frac{E_y \Phi}{1 - \nu_{xy} \nu_{yx}} \left(\frac{1}{G_{yz}} \frac{\partial^2 \varphi}{\partial y^2} + \frac{\nu_{xy}}{G_{xz}} \frac{\partial^2 \varphi}{\partial x^2} \right) + A_y \sigma_z; \\ \tau_{xy} &= G_{xy} \left[-2z \frac{\partial^2 w}{\partial x \partial y} - \gamma_{xy}^* + \Gamma_{xy} + \Phi(z) \frac{\partial^2 \varphi}{\partial x \partial y} \left(\frac{1}{G_{xz}} + \frac{1}{G_{yz}} \right) \right]. \end{aligned} \quad (17)$$

The bending and twisting moment will be written in the form:

$$\begin{aligned}
M_x &= \int_{-h/2}^{h/2} \sigma_x z dz = -D_x \left(\frac{\partial^2 w}{\partial x^2} + \nu_{yx} \frac{\partial^2 w}{\partial y^2} \right) - M_x^* + \frac{4}{5} D_x \left(\frac{1}{G_{xz}} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\nu_{yx}}{G_{yz}} \frac{\partial^2 \varphi}{\partial y^2} \right) + \frac{qh^2}{10} A_x; \\
M_y &= \int_{-h/2}^{h/2} \sigma_y z dz = -D_y \left(\frac{\partial^2 w}{\partial y^2} + \nu_{xy} \frac{\partial^2 w}{\partial x^2} \right) - M_y^* + \frac{4}{5} D_y \left(\frac{1}{G_{yz}} \frac{\partial^2 \varphi}{\partial y^2} + \frac{\nu_{xy}}{G_{xz}} \frac{\partial^2 \varphi}{\partial x^2} \right) + \frac{qh^2}{10} A_y; \quad (18) \\
H &= \int_{-h/2}^{h/2} \tau_{xy} z dz = -2D_K \frac{\partial^2 w}{\partial x \partial y} + \frac{4}{5} D_K \left(\frac{1}{G_{xz}} + \frac{1}{G_{yz}} \right) \frac{\partial^2 \varphi}{\partial x \partial y} - H^*,
\end{aligned}$$

$$\text{where } M_x^* = \frac{E_x}{1 - \nu_{xy} \nu_{yx}} \int_{-h/2}^{h/2} [\varepsilon_x^* + \nu_{yx} \varepsilon_y^* - (\Gamma_x + \nu_{yx} \Gamma_y)] z dz; D_x = \frac{E_x h^3}{12(1 - \nu_{xy} \nu_{yx})};$$

$$M_y^* = \frac{E_y}{1 - \nu_{xy} \nu_{yx}} \int_{-h/2}^{h/2} [\varepsilon_y^* + \nu_{xy} \varepsilon_x^* - (\Gamma_y + \nu_{xy} \Gamma_x)] z dz; D_y = \frac{E_y h^3}{12(1 - \nu_{xy} \nu_{yx})};$$

$$H^* = G_{xy} \int_{-h/2}^{h/2} (\gamma_{xy}^* - \Gamma_{xy}) z dz; D_K = \frac{G_{xy} h^3}{12(1 - \nu_{xy} \nu_{yx})}.$$

The equilibrium equation of the plate is written as:

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q(x, y). \quad (19)$$

Substituting (18) into (19), we obtain the following equation:

$$\begin{aligned}
D_x \frac{\partial^4 w}{\partial x^4} + 2D_0 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} &= \frac{4}{5} \left[\frac{D_x}{G_{xz}} \frac{\partial^4 \varphi}{\partial x^4} + D_0 \left(\frac{1}{G_{xz}} + \frac{1}{G_{yz}} \right) \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{D_y}{G_{yz}} \frac{\partial^4 \varphi}{\partial y^4} \right] + \\
&+ \frac{h^2}{10} \left(A_x \frac{\partial^2 q}{\partial x^2} + A_y \frac{\partial^2 q}{\partial y^2} \right) + q(x, y) + q^*(x, y), \quad (20)
\end{aligned}$$

$$\text{where } q^*(x, y) = - \left(\frac{\partial^2 M_x^*}{\partial x^2} + 2 \frac{\partial^2 H^*}{\partial x \partial y} + \frac{\partial^2 M_y^*}{\partial y^2} \right).$$

Thus, the problem is reduced to a system of two differential equations (9) and (20) with respect to the shift and deflection function.

To determine the creep strains, at the first step we solve the elastic problem, then the grid in time is introduced and the Euler method is used.

For a rectangular hinged on the contour plate, the functions $\varphi(x, y)$ и $w(x, y)$ can be represented in the form of double trigonometric series:

$$\varphi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \quad w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (21)$$

This solution exactly satisfies the boundary conditions at constraints on the edges of tangential displacements:

$$w = M_x = v = 0 \text{ at } x = \text{const}; \quad w = M_y = u = 0 \text{ at } y = \text{const}. \quad (22)$$

The functions $q(x, y)$ and $q^*(x, y)$ can be expanded in a Fourier series by the formulas:

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \quad q^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn}^* \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (23)$$

The coefficients of the expansion are determined by the formulas:

$$q_{mn} = \frac{4}{ab} \int_0^a \int_0^b q(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy; \quad q_{mn}^* = \frac{4}{ab} \int_0^a \int_0^b q^*(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy. \quad (24)$$

The second integral in (24) is calculated numerically.

Substituting the expansions of the functions $q(x, y)$ and $\varphi(x, y)$ into equation (9), we obtain:

$$\varphi_{mn} = 3q_{mn} / \left\{ 2h \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] \right\}. \quad (25)$$

It can be seen from equation (9) that the shift function is constant in time. After substituting the expansions of the functions $w(x, y)$, $\varphi(x, y)$, $q(x, y)$, $q^*(x, y)$ into equation (20) for each of the coefficients w_{mn} , we obtain the following equality:

$$a_{11} w_{mn} = a_{1p}, \quad (26)$$

$$\begin{aligned} \text{where } a_{11} &= D_x \left(\frac{m\pi}{a} \right)^4 + 2D_0 \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 + D_y \left(\frac{n\pi}{b} \right)^4; \\ a_{1p} &= q_{mn} \left(1 - \frac{h^2}{10} \left[A_x \left(\frac{m\pi}{a} \right)^2 + A_y \left(\frac{n\pi}{b} \right)^2 \right] \right) + q_{mn}^* + \frac{4}{5} \varphi_{mn} \left(\frac{D_x}{G_{xz}} \left(\frac{m\pi}{a} \right)^4 + \left(\frac{n\pi}{b} \right)^4 \frac{D_y}{G_{yz}} + \right. \\ &\quad \left. + \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 D_0 \left(\frac{1}{G_{xz}} + \frac{1}{G_{yz}} \right) \right). \end{aligned}$$

3 Results

A test problem was solved for a hingedly supported plate of unidirectional fiberglass loaded with a uniformly distributed load. Elastic parameters of the material:

$$E_x = 4.95 \cdot 10^4 \text{ MPa}, \quad E_y = E_z = 8.62 \cdot 10^3 \text{ MPa}, \quad \nu_{xy} = \nu_{xz} = 0.3366, \quad \nu_{yx} = \nu_{zx} = 0.0586,$$

$\nu_{yz} = 0.3, G_{xz} = G_{yz} = G_{xy} = 2.53 \cdot 10^3 \text{ MPa}$. As the law of creep we used the following equation:

$$\varepsilon_{ij}(t) = I_{ijkl} \sigma_{kl}(t) + \int_0^t K_{ijkl}(t-\tau) \sigma_{kl}(\tau) d\tau, \quad (27)$$

where σ_{kl} is the stress tensor, ε_{ij} is the strain tensor, $K_{ijkl}(t-\tau)$ are the functions of the creep kernels, which were taken as a sum of exponentials.

Rheological parameters of the material were taken from work [5]. Plate dimensions: $a = 1.5 \text{ m}$, $b = 2 \text{ m}$, $h = 6 \text{ cm}$. The load value was $q = 1 \text{ kPa}$.

The resulting curve of the deflection growth is shown in Figure 1. The dashed line corresponds to a solution based on the Kirchhoff-Love theory. When the shear deformations γ_{xz} and γ_{yz} are taken into account, the deflection at the beginning of the process turns out to be higher by 2.35%, and at $t \rightarrow \infty$ – by 2.2%.

Fig. 1. The graph of the deflection growth.

The time variation of the maximum stress σ_x is shown in Figure 2. It can be seen from the presented graph that when calculating with the corrected theory, the stresses in the plate are higher in comparison with the Kirchhoff-Love theory.

Fig. 2. The time variation of the maximum stress σ_x

4 Discussion

It is known that for isotropic plates the classical theory of plates is applicable if the following relation holds [6]:

$$\frac{1}{80} \leq \frac{h}{a} \leq \frac{1}{5},$$

where a is the smallest plate size in plan.

In spite of the fact that the ratio h/a for the considered plate is $1/25$, the results are already deviating from the classical theory because of its pronounced anisotropy.

The equations given in this paper are universal and allow us to use an arbitrary law of creep. In the case of arbitrary fastening of the plate edges, the solution of the system of differential equations can be performed numerically using the finite difference method.

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