

Comparison of the stress and strain intensity factors for the corner area of the structure boundary

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Abstract. The stress-strain state of structures in areas with corner cut-outs and cuts of boundaries features the occurrence of areas of stress concentration and requires assessment of strength and reliability of facilities, which is a relevant task in engineering practice. Theoretical analysis of stress-strain state (SSS) of corner cut-outs zones of the area boundary is reduced to the study of singular solutions of the elasticity theory problem with exponential features. At that, the concept of stress or strain concentration in an irregular point of the area boundary is meaningless. This paper considers the stress-strain state in the vicinity of the top of the corner cut-out of the flat area boundary, which is recorded using the intensity factors as limit values of stresses and strains. We give two approaches for obtaining the limit values for stress and strain in the vicinity of an irregular point of the plane area boundary using the stress intensity factors and the strain intensity factors. The stress-strain state in the corner cut-outs zone of structures and buildings boundary recorded in the form of limit values of stresses and strains may further be used to determine and record the influence of changing the factors of intensity of stresses and strains on SSS of structures, which is a separate task of solid mechanics. The difference in the expressions of stresses and displacements obtained for limit values of stresses and strains determines practical significance of the work when carrying out experiments and at determination of critical values of stresses and strains.

1 Introduction

The stress-strain state (SSS) of composite structures in zones of connection of elements under the action of forced strains being ruptural along the line (surface) of the elements contact is characterized by exponential feature. The stress-strain state in the irregular point zone of the area boundary under the action of forced strains, in particular temperature ones, is determined by solution of the homogeneous boundary value problem of elasticity theory. Solution of the homogeneous elasticity problem in the vicinity of an irregular point on a particular line is reduced [1-4] to the solution of the problem of plane strain: $\sigma_x \neq 0$, $\sigma_y \neq 0$

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, $\tau_{xy} \neq 0$, $\tau_{xz} = \tau_{yz} = 0$, $\frac{\partial}{\partial z} \approx 0$, $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ and the problem of anti-plane strain:
 $\varepsilon_{xz} \neq 0$; $\varepsilon_{yz} \neq 0$; $W \neq 0$; $\frac{\partial}{\partial z} \approx 0$; $n_z = 0$, $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{xy} = 0$, $U = V = 0$. Solution of plane problems in the polar coordinate system is searched by separation of variables [3-6].

The exponential feature of the SSS depends on the shape of the area boundary - a cut-out, a cut, the order of which depends on the eigenvalues of the homogeneous elastic problem [4-6]. Eigenvalues of the homogeneous boundary value problem depend on the shape of boundary, type of boundary conditions, mechanical characteristics of the area material and have an array of values [5,8-10]. In the fracture mechanics, for a perfect mathematical cut, singular stresses are characterized by the stress intensity factors [3,5,7,8,11-18]. The purpose of this work is to determine the stress-strain state in the zone of corner cut-out of arbitrary opening of the area boundary using factors of intensity of stresses and strains as the limit values of stresses and strains.

2 Method of solving the problem

2.1 Problem formulation. Plane strain case

We consider here a homogeneous boundary value problem of elasticity theory for a flat wedge-shaped area with a symmetrical opening 2α in the vicinity of the corner point of the boundary - a top of the corner cut-out of an arbitrary opening [3,5,11-13]. The solution of the problem of plane strain when moving in polar coordinates system is searched as follows:

$$u_r(r, \theta) = r^\lambda f(\theta), \quad u_\theta(r, \theta) = r^\lambda g(\theta) \quad (1)$$

where $f(\theta)$, $g(\theta)$ are unknown functions of angle θ to be determined, λ is the unknown parameter. If we put relations (1) to the Lamé equations, we obtain expressions for displacements and stresses in the vicinity of an irregular point of the area boundary [3,5,13] containing four unknown factors A, B, C, D.

When meeting homogeneous boundary conditions: $\sigma_\theta = \tau_{r\theta} = 0$ at $\theta = \pm\alpha$, we get two homogeneous systems of linear equations relative to the unknown constants A, B, C, D. Determinants of the systems have the following form:

$$\lambda \sin 2\alpha + \sin 2\lambda\alpha = 0, \quad (2a)$$

$$-\lambda \sin 2\alpha + \sin 2\lambda\alpha = 0. \quad (2b)$$

Let λ_i^- stand for the system of roots of the equation (2a). SSS in the area of corner cut-out of area boundary containing factors A, C is recorded using roots λ_i^- .

Let λ_i^+ stand for the system of roots of the equation (2b). SSS in the area of corner cut-out of area boundary containing factors B, D is recorded using roots λ_i^+ .

Subject to the conditions (2a), we obtain relations for factors A and C:

$$A = \frac{(1 - \lambda^-) \sin \left[(1 - \lambda^-)\alpha \right]}{(k - \lambda^-) \sin \left[(1 + \lambda^-)\alpha \right]} C, \quad (3)$$

Subject to the conditions (2b), we obtain relations for factors B and D:

$$B = \frac{(1 + \lambda^+) \sin[(1 - \lambda^+) \alpha]}{(k - \lambda^+) \sin[(1 + \lambda^+) \alpha]} D, \tag{4}$$

where $k = 3 - 4\nu$, ν is Poisson factor of the area material, λ are eigenvalues of homogeneous boundary-value problem of elasticity theory in the zone of the corner cut-out of the area boundary, in general case, complex numbers, $\lambda^- = \min \operatorname{Re} \lambda_i^-$, $\lambda^+ = \min \operatorname{Re} \lambda_i^+$ are the minimum values of real parts of complex roots of the characteristic equations (2a), (2b), respectively.

SSS in the vicinity of the top of the corner cut-out of the area boundary taking into account the differences of the roots of the characteristic equations (2a), (2b) will be written as follows:

$$\begin{aligned} u_r &= r^{\lambda^-} \{A \cos[(1 + \lambda^-) \theta] + C \cos[(1 - \lambda^-) \theta]\} + \\ &+ r^{-\lambda^+} \{B \sin[(1 + \lambda^+) \theta] + D \sin[(1 - \lambda^+) \theta]\}, \\ u_\theta &= r^{\lambda^-} \{-A \sin[(1 + \lambda^-) \theta] - \nu_2^- C \sin[(1 - \lambda^-) \theta]\} + \\ &+ r^{-\lambda^+} \{B \cos[(1 + \lambda^+) \theta] + \nu_2^+ D \cos[(1 - \lambda^+) \theta]\}, \\ \sigma_\theta &= \mu r^{\lambda^- - 1} \{-2\lambda^- A \cos[(1 + \lambda^-) \theta] - (1 + \lambda^-)(1 - \nu_2^-) C \cos[(1 - \lambda^-) \theta]\} + \\ &+ \mu r^{\lambda^+ - 1} \{-2\lambda^+ B \sin[(1 + \lambda^+) \theta] - (1 + \lambda^+)(1 - \nu_2^+) D \sin[(1 - \lambda^+) \theta]\}, \\ \tau_{r\theta} &= \mu r^{\lambda^- - 1} \{-2\lambda^- A \sin[(1 + \lambda^-) \theta] - (1 - \lambda^-)(1 - \nu_2^-) C \sin[(1 - \lambda^-) \theta]\} + \\ &+ \mu r^{\lambda^+ - 1} \{2\lambda^+ B \cos[(1 + \lambda^+) \theta] + (1 - \lambda^+)(1 - \nu_2^+) D \cos[(1 - \lambda^+) \theta]\}, \\ \sigma_r &= 2\mu \lambda r^{\lambda^- - 1} \{(A \cos[(1 + \lambda^-) \theta] + \frac{3 - \lambda^-}{k - \lambda^-} C \cos[(1 - \lambda^-) \theta])\} + \\ &+ 2\mu \lambda r^{\lambda^+ - 1} \{(B \sin[(1 + \lambda^+) \theta] + \frac{3 - \lambda^+}{k - \lambda^+} D \sin[(1 - \lambda^+) \theta])\} \end{aligned} \tag{5}$$

where ν, μ is Poisson factor and shear modulus, respectively, $\nu_2^- = \frac{3 + \lambda_2^- - 4\nu}{3 - \lambda_2^- - 4\nu}$,

$\nu_2^+ = \frac{3 + \lambda_2^+ - 4\nu}{3 - \lambda_2^+ - 4\nu}$, $\lambda^- = \min \operatorname{Re} \lambda_i^-$, $\lambda^+ = \min \operatorname{Re} \lambda_i^+$. The unknown factors A, B, C, D -

arbitrary constants to be determined - satisfy relations (3), (4).

Let's consider SSS in the form of (5) with the help of limit values of stresses and strains in the vicinity of the top of the corner cut-out of the boundary of arbitrary opening area called the stress and strain intensity factors, like the stress and strain intensity factors in fracture mechanics [3,8,12,13].

2.2 Area stress intensity factors

For stresses σ_θ in expression (5), let's proceed to the limit at $\theta = 0$:

$$\lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^-} \sigma_{\theta, \theta=0} = -2A\lambda^- - (1+\lambda^-)(1-\nu_2^-)C. \quad (6)$$

Let's denote limit values of stresses (6) as follows:

$$K_I^\sigma = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^-} \sigma_{\theta, \theta=0}. \quad (7)$$

Factors A and C taking into account relations (3) and designations (7) will be written as follows:

$$A = \frac{(1-\lambda^-) \sin[(1-\lambda^-)\alpha]}{2\lambda^- \left[(\lambda^- - 1) \sin[(1-\lambda^-)\alpha] + (\lambda^- + 1) \sin[(1+\lambda^-)\alpha] \right]} K_I^\sigma, \quad (8)$$

$$C = \frac{(k-\lambda^-) \sin[(1+\lambda^-)\alpha]}{2\lambda^- \left[(\lambda^- - 1) \sin[(1-\lambda^-)\alpha] + (\lambda^- + 1) \sin[(1+\lambda^-)\alpha] \right]} K_I^\sigma. \quad (9)$$

To find factors B and D in expression (5) for stresses $\tau_{r\theta}$, let's proceed to the limit at $\theta = 0$:

$$\lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^+} \tau_{r\theta, \theta=0} = 2\lambda^+ B + (1-\lambda^+)(1-\nu_2^+) D. \quad (10)$$

Let's denote limit values of stresses (10) as follows:

$$K_{II}^\sigma = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^+} \tau_{r\theta, \theta=0}. \quad (11)$$

Factors B and D taking into account relations (4) and designations (11) will be written as follows:

$$B = \frac{(1+\lambda^+) \sin[(1-\lambda^+)\alpha]}{2\lambda^+ \left[(\lambda^+ + 1) \sin[(1-\lambda^+)\alpha] - (1-\lambda^+) \sin[(1+\lambda^+)\alpha] \right]} K_{II}^\sigma, \quad (12)$$

$$D = \frac{(k-\lambda^+) \sin[(1+\lambda^+)\alpha]}{2\lambda^+ \left[(\lambda^+ + 1) \sin[(1-\lambda^+)\alpha] - (1-\lambda^+) \sin[(1+\lambda^+)\alpha] \right]} K_{II}^\sigma. \quad (13)$$

Factors $K_I^\sigma = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^-} \sigma_{\theta, \theta=0}$, $K_{II}^\sigma = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^+} \tau_{r\theta, \theta=0}$ in the form of (7), (11) respectively, in the same way as designations accepted in fracture mechanics [3,4,13-16], are defined as the stress intensity factors, values $\lambda^- = \min \operatorname{Re} \lambda_i^-$, $\lambda^+ = \min \operatorname{Re} \lambda_i^+$ are solutions of characteristic equations (2a), (2b), respectively.

Taking into account found constants A, B, C, D in the form of (8), (9), (12), (13) and designation of limit values of stresses K_I^σ , K_{II}^σ (stress intensity factors) in the form of (7), (11), the initial stress - strain state (5) in the zone of corner cut-out of arbitrary opening of the area boundary will be written as a function of the stress intensity factors.

2.3. The area strain intensity factors

Let's write the expression for strain ε_0 in the zone of corner cut of the area border:

$$\begin{aligned} \varepsilon_{\theta} = r^{\lambda^{-}-1} \{ -A \lambda^{-} \cos \left[(1 + \lambda^{-}) \theta \right] + \frac{(1 + \lambda^{-} - 4\nu)}{k - \nu} C \lambda^{-} \cos \left[(1 - \lambda^{-}) \theta \right] \} + \\ + r^{\lambda^{+}-1} \{ -B \lambda^{+} \sin \left[(1 + \lambda^{+}) \theta \right] + \frac{(1 + \lambda^{+} - 4\nu)}{k - \nu} D \lambda^{+} \sin \left[(1 - \lambda^{+}) \theta \right] \} \end{aligned} \quad (14)$$

To find factors A and C in expression (12) for strains ε_{θ} , let's proceed to the limit at $\theta = 0$:

$$\lim_{r \rightarrow 0} r^{1-\lambda^{-}} \varepsilon_{\theta, \theta=0} = \lambda^{-} \left[-A + \frac{(1 + \lambda^{-} - 4\nu)}{k - \nu} C \right]. \quad (15)$$

Let's denote limit values of strains (15) as follows:

$$K_I^{\varepsilon} = \lim_{r \rightarrow 0} r^{1-\lambda^{-}} \varepsilon_{\theta, \theta=0}. \quad (16)$$

Factors A and C taking into account relations (3) and designations (16) will be written as follows:

$$C = \frac{(k - \lambda^{-}) \sin \left[(1 + \lambda^{-}) \alpha \right]}{\lambda^{-} \left((1 + \lambda^{-} - 4\nu) \sin \left[(1 + \lambda^{-}) \alpha \right] - (1 - \lambda^{-}) \sin \left[(1 - \lambda^{-}) \alpha \right] \right)} K_I^{\varepsilon}, \quad (17)$$

$$A = \frac{(1 - \lambda^{-}) \sin \left[(1 - \lambda^{-}) \alpha \right]}{\lambda^{-} \left((1 + \lambda^{-} - 4\nu) \sin \left[(1 + \lambda^{-}) \alpha \right] - (1 - \lambda^{-}) \sin \left[(1 - \lambda^{-}) \alpha \right] \right)} K_I^{\varepsilon}. \quad (18)$$

Let's write the expression for strain $\varepsilon_{r\theta}$ in the zone of corner cut of the area border:

$$\begin{aligned} \varepsilon_{r\theta} = r^{\lambda^{-}-1} \{ -2\lambda^{-} A \sin \left[(1 + \lambda^{-}) \theta \right] - (1 - \lambda^{-}) (1 - \nu_2^{-}) C \sin \left[(1 - \lambda^{-}) \theta \right] + \\ + r^{\lambda^{+}-1} \{ 2\lambda^{+} B \cos \left[(1 + \lambda^{+}) \theta \right] + (1 - \lambda^{+}) (1 - \nu_2^{+}) D \cos \left[(1 - \lambda^{+}) \theta \right] \} \end{aligned} \quad (19)$$

To find factors B and D in expression (19) for strains $\varepsilon_{r\theta}$, let's proceed to the limit at $\theta = 0$:

$$\lim_{r \rightarrow 0} r^{1-\lambda^{+}} \varepsilon_{r\theta, \theta=0} = 2\lambda^{+} B + (1 - \lambda^{+}) (1 - \nu_2^{+}) D. \quad (20)$$

Let's denote limit values of strains (20) as follows:

$$K_{II}^{\varepsilon} = \lim_{r \rightarrow 0} r^{1-\lambda^{+}} \varepsilon_{r\theta, \theta=0}. \quad (21)$$

Taking into account proportion $\varepsilon_{r\theta} = \mu^{-1} \tau_{r\theta}$, we get the following:

$$K_{II}^{\varepsilon} = \lim_{r \rightarrow 0} r^{1-\lambda^{+}} \varepsilon_{r\theta, \theta=0} = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda^{+}} \tau_{r\theta, \theta=0} = K_{II}^{\sigma}, \text{ i.e. } K_{II}^{\sigma} = K_{II}^{\varepsilon},$$

therefore, factors B and D have the form similar to (12), (13):

$$B = \frac{(1 + \lambda^{+}) \sin \left[(1 - \lambda^{+}) \alpha \right]}{2\lambda^{+} \left[(\lambda^{+} + 1) \sin \left[(1 - \lambda^{+}) \alpha \right] - (1 - \lambda^{+}) \sin \left[(1 + \lambda^{+}) \alpha \right] \right]} K_{II}^{\varepsilon}, \quad (22)$$

$$D = \frac{(k-\lambda^+) \sin \left[(1+\lambda^+) \alpha \right]}{2\lambda^+ \left[(\lambda^+ + 1) \sin \left[(1-\lambda^+) \alpha \right] - (1-\lambda^+) \sin \left[(1+\lambda^+) \alpha \right] \right]} K_{II}^\varepsilon. \quad (23)$$

Factors $K_I^\varepsilon = \lim_{r \rightarrow 0} r^{1-\lambda^-} \varepsilon_{\theta, \theta=0}$, $K_{II}^\varepsilon = \lim_{r \rightarrow 0} r^{1-\lambda^+} \varepsilon_{r\theta, \theta=0}$ in the form of (16), (21) in the same way as designations accepted in fracture mechanics [3,8,13-15] are defined as strain intensity factors. Values $\lambda^- = \min \operatorname{Re} \lambda_i^-$, $\lambda^+ = \min \operatorname{Re} \lambda_i^+$ are solutions of characteristic equations (2a), (2b), respectively.

Taking into account found factors A, B, C, D in the form of (17), (18), (22), (23) and designations of limit values of strains K_I^ε , K_{II}^ε (strain intensity factors) in the form of (16), (21), the initial stress - strain state (5) in the zone of corner cut of an arbitrary opening of the area border will be written as functions of strain intensity factors.

For the case of anti-plane strain task, in a similar way we write a solution in the vicinity of an irregular point of the area boundary in the following form:

$$w = r^{\lambda_1} [D_1 \sin \lambda_1 \theta + D_2 \cos \lambda_1 \theta]$$

$$\mu^{-1} r^{1-\lambda_1} \tau_{xz} = \lambda_1 \{-D_1 \sin [(1-\lambda_1)\theta] + D_2 \cos [(1-\lambda_1)\theta]\}, \quad (24)$$

$$\mu^{-1} r^{1-\lambda_1} \tau_{yz} = \lambda_1 \{D_1 \cos [(1-\lambda_1)\theta] + D_2 \sin [(1-\lambda_1)\theta]\},$$

where D_1, D_2 are arbitrary constants to be determined, where λ_1 are eigenvalues of the homogeneous boundary value problem.

After identifying as follows:

$$K_{III}^1 = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda_1} \tau_{yz, \theta=0}, \quad K_{III}^2 = \lim_{r \rightarrow 0} \mu^{-1} r^{1-\lambda_1} \tau_{xz, \theta=0}, \quad (25)$$

we write factors D_1 and D_2 , at that stresses and strains intensity factors numerically are the same.

3 Results

A method of determining the stress-strain state in the corner cut zone of the plane area boundary using limit values of stress and strain called stress and strain intensity factors, respectively, is given. For the SSS in the form of (5) in the corner cut zone of the area boundary, the unknown factors A, B, C, D in the form of (8), (9), (12), (13) have been defined using limit values of stresses K_I^σ , K_{II}^σ in the form of (7), (11). For the SSS in the form of (5) in the corner cut zone of the area boundary, the unknown factors A, B, C, D in the form of (17), (18), (22), (23) have been defined using limit values of strains K_I^ε , K_{II}^ε in the form of (16),(21).

4 Discussion

The stress and strain intensity factors for normal stresses and linear strains are different. Intensity factors for shear stresses and angular strains are the same. In the general case of the stress - strain state in the vicinity of the top of the cut of the area boundary there is no proportionality of the stresses and strains intensity factors, which shall be taken into account in experimental determination of their values.

5 Conclusions

The stress-strain state in the corner cut-outs zone of structures and buildings boundary recorded in the form of limit values of stresses and strains further may be used to determine and record the influence of changing the factors of intensity of stresses and strains on SSS of structures, which is a separate task of solid mechanics.

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