Field theory in normal toroidal coordinates

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Abstract. Field theory is widely represented in spherical and cylindrical coordinate systems, since the mathematical apparatus of these coordinate systems has been thoroughly studied. Sources of field with more complex structures require new approaches to their study. The purpose of this research is to adapt the field theory referred to curvilinear coordinates and represent it in normal toroidal coordinates. Another purpose is to develop the foundations of geometric modeling with the use of computer graphics for visualizing the level surfaces. The dependence of normal toroidal coordinates on rectangular Cartesian coordinates and Lamé coefficients is shown in this scientific paper. Differential characteristics of scalar and vector fields in normal toroidal coordinates are obtained: scalar and vector field laplacians, divergence, and rotation of vector field. The example shows the technique of modeling the field and its further computer visualization. The technique of reading the internal equation of the surface is presented and the influence of the values of the parameters on the shape of the surface is shown. For the first time, expressions of scalar and vector field characteristics in normal toroidal coordinates are obtained, the fundamentals of geometric modeling of fields with the use of computer graphics tools are developed for the purpose of providing visibility for their study.

1 Introduction

Most physical processes and phenomena are modeled using the mathematical apparatus of field theory. The field theory in the scientific literature is widely represented in a vector notation in rectangular Cartesian coordinates [1, 2].

If the field source is concentrated at a point, its characteristics are more convenient to describe in spherical coordinates. The isosurfaces (level surfaces) of such a field are concentric spheres with the center at the source of the field. If the field source is distributed along a straight line, its characteristics are obtained in cylindrical coordinates. The level surfaces of such a field are coaxial cylinders.

It is fields with a point and linear sources that are thoroughly studied and covered in the scientific literature [3]. Since the mathematical apparatus is based in this case on the use of the simplest, after Cartesian, classical coordinate systems - spherical and cylindrical.

Complications of the mathematical plan, arising in the description of fields of complex structure, for example, fields with a source of more complex shape than the point and the line, require new approaches to the study of processes occurring in such fields. Especially

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this applies to cases where a certain object of a complicated form is simultaneously a sink of the action of one field and the source of the other [4, 5].

In addition to cylinders and spheres, following by simplicity, elementary surfaces are the cone and torus. The surfaces of the level of the thermal field with the source in the form of external conical and torus surfaces do not coincide with the families of equidistant cones and tori. These families are not coordinate in one of the well-known coordinate systems. This is the cause of the complications that arise when describing fields and determining their characteristics.

The main idea of the work is to introduce, represent the field and determine its characteristics, new or well-known coordinate systems. In this case, the condition of coincidence of the coordinate families of surfaces with the level surfaces of the field or with the limiting surfaces of the process must be kept.

2 Methods

Functions that enter toroidal coordinates:

$$x = (R + (r + v) \cdot \cos u) \cos t, \quad y = (R + (r + v) \cdot \cos u) \sin t, \quad z = (r + v) \cdot \sin u \quad (1)$$

Area of correct coordination by normal toroidal coordinates

$$\frac{D(x, y, z)}{D(t, u, v)} = \frac{\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}} = (r + v)[R + (r + v) \cos u] \neq 0 \quad (2)$$

It follows from (2) that the correctness of the coordination of space is violated when $v = -\frac{R}{\cos u} - r$, that is, at points located on the $0z$ axis, as well as at $v = -r$, which corresponds to the points on the centerline of the generating circle of the determinant torus. It is to be recalled that R is the radius of the centers of the generating circle; r is the radius of the generating circle of the torus surface ($R > r$).

Functions of the dependence of the normal toroidal coordinates t, u, v on rectangular Cartesian coordinates:

$$\begin{aligned} x = 0, \quad y > 0 &\Rightarrow t = \frac{\pi}{2}; \\ x = 0, \quad y < 0 &\Rightarrow t = \frac{3\pi}{2}; \\ x > 0, \quad y > 0 &\Rightarrow t = \arctg \frac{y}{x}; \\ x < 0, &\quad t = \arctg \frac{y}{x} + \pi; \\ x > 0, \quad y < 0 &\Rightarrow t = \arctg \frac{y}{x} + 2\pi; \\ u = \arctg \frac{z}{\sqrt{x^2 + y^2} - R}; &\quad v = \sqrt{(\sqrt{x^2 + y^2} - R)^2 + z^2} - r. \end{aligned} \quad (3)$$

Obtained for the Lamé coefficients:

$$\begin{aligned} H_t &= \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} = R + (r + v)\cos u, \\ H_u &= \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} = r + v, \\ H_v &= \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2} = 1. \end{aligned} \tag{4}$$

The system of normal toroidal coordinates is a triorthogonal system, that is, the pairs of its coordinate surfaces ($t = const$ - semiplane, $u = const$ - cones of rotation, $v = const$ - tori) intersect at the lines of curvature.

Both the global (x, y, z) and local (t, u, v) bases (Fig. 1) are right (clockwise).

Fig. 1. The image of the determinant torus

Differential characteristics of scalar and vector fields in normal toroidal coordinates.
The scalar field derivative in the direction

$$\vec{l}^0 = \cos\alpha e_t + \cos\beta e_u + \cos\gamma e_v$$

is determined by the formula

$$\frac{dF}{dl} = \frac{G_t}{|G|} \cos\alpha + \frac{G_u}{|G|} \cos\beta + \frac{G_v}{|G|} \cos\gamma, \tag{5}$$

where $G_t = \frac{1}{R + (r + v)\cos u} \frac{\partial F}{\partial t}$, $G_u = \frac{1}{r + v} \frac{\partial F}{\partial u}$, $G_v = \frac{\partial F}{\partial v}$, $G = \sqrt{G_t^2 + G_u^2 + G_v^2}$

Laplacian of a scalar field

$$\begin{aligned} \Delta F &= \frac{1}{[R + (r + v)\cos u]^2} \frac{\partial^2 F}{\partial t^2} + \frac{1}{(r + v)^2} \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} + \\ &+ \frac{1}{(r + v)[R + (r + v)\cos u]} \left\{ [R + 2(r + v)\cos u] \frac{\partial F}{\partial v} - \sin u \frac{\partial F}{\partial u} \right\}. \end{aligned} \tag{6}$$

Divergence of a vector field

$$\operatorname{div} \bar{a} = \frac{1}{R + (r + v) \cos u} \frac{\partial a_t}{\partial t} + \frac{1}{r + v} \frac{\partial a_u}{\partial u} + \frac{\partial a_v}{\partial v} - \frac{a_u \sin u}{R + (r + v) \cos u} + \frac{a_v [R + 2(r + v) \cos u]}{[R + (r + v) \cos u](r + v)}. \quad (7)$$

Laplacian of a vector field

$$\begin{aligned} \Delta \bar{a} = & \left\{ \frac{1}{[R + (r + v) \cos u]^2} \frac{\partial^2 a_t}{\partial t^2} + \frac{1}{(r + v)^2} \frac{\partial^2 a_t}{\partial u^2} + \frac{\partial^2 a_t}{\partial v^2} + \frac{1}{[R + (r + v) \cos u](r + v)} \times \right. \\ & \times \left[(R + 2(r + v) \cos u) \frac{\partial a_t}{\partial v} - \sin u \frac{\partial a_t}{\partial u} \right] \Big\} e_t + \left\{ \frac{1}{[R + (r + v) \cos u]^2} \frac{\partial^2 a_u}{\partial t^2} + \right. \\ & + \frac{1}{(r + v)^2} \frac{\partial^2 a_u}{\partial u^2} + \frac{\partial^2 a_u}{\partial v^2} + \frac{1}{[R + (r + v) \cos u](r + v)} \left[(R + 2(r + v) \cos u) \frac{\partial a_u}{\partial v} - \right. \\ & \left. \left. - \sin u \frac{\partial a_u}{\partial u} \right] \right\} e_u + \left\{ \frac{1}{[R + (r + v) \cos u]^2} \frac{\partial^2 a_v}{\partial t^2} + \frac{1}{(r + v)^2} \frac{\partial^2 a_v}{\partial u^2} + \frac{\partial^2 a_v}{\partial v^2} + \right. \\ & \left. + \frac{1}{[R + (r + v) \cos u](r + v)} \left[(R + 2(r + v) \cos u) \frac{\partial a_v}{\partial v} - \sin u \frac{\partial a_v}{\partial u} \right] \right\} e_v. \end{aligned} \quad (8)$$

Rotor (curl) of a vector field

$$\begin{aligned} \operatorname{rot} \bar{a} = & \left[\frac{1}{r + v} \left(\frac{\partial a_v}{\partial u} - a_u \right) - \frac{\partial a_u}{\partial v} \right] e_t + \left[\frac{1}{R + (r + v) \cos u} \left(a_t \cos u - \frac{\partial a_v}{\partial t} \right) + \frac{\partial a_t}{\partial v} \right] e_u + \\ & + \left[\frac{1}{R + (r + v) \cos u} \left(\frac{\partial a_u}{\partial t} + a_t \sin u \right) - \frac{1}{r + v} \frac{\partial a_t}{\partial u} \right] e_v. \end{aligned} \quad (9)$$

The presented expressions for the differential characteristics of scalar and vector fields greatly simplify their representation and study in normal toroidal coordinates.

Examples of numerical and graphical study of fields represented in normal toroidal coordinates

Let us show the application of the reduced mathematical apparatus for geometric modeling of fields represented in normal toroidal coordinates.

Example. Calculate the derivative of the scalar field

$$F = h \sin u \cos nt - v, \quad (10)$$

represented in normal toroidal coordinates (1), at $h = 0.75$, $n = 6$, $R = 5$, $r = 2$ in the direction

$$\vec{l}^\circ = 0.6e_t + 0.8e_v \quad (11)$$

at the point $t = 0.6$, $u = 2.1$, $v = 0.5$. Provide an image of level surfaces

$$v = h \sin u \cos nt + i\Delta C \quad (12)$$

at $i = -1, 0, 1$, $\Delta C = 0.5$. Give the method of reading the intrinsic equation (12) of the level surface and show the influence of the values of the parameters on the shape of its surface.

Solution. Let us calculate the local coordinates of the gradient vector of the function (10) by the formulas (5) at a given point

$$\begin{aligned} G_t &= \frac{1}{H_t} \frac{\partial F}{\partial t} = \frac{1}{R + (r + v) \cos u} (-nh \sin u \sin nt) = \\ &= \frac{1}{5 + (2 + 0.5) \cos 2.1} (-6 \cdot 0.75 \sin 2.1 \cdot \sin(6 \cdot 0.6)) = -0.3661, \end{aligned}$$

$$\begin{aligned}
 G_u &= \frac{1}{H_v} \frac{\partial F}{\partial u} = \frac{1}{r+v} h \sin u \cos nt = \\
 &= \frac{1}{2+0,5} 0,75 \cos 2,1 \cdot \cos(6 \cdot 0,6) = -0,1417, \\
 G_v &= \frac{1}{H_v} \frac{\partial F}{\partial v} = \frac{1}{1} (-1) = -1.
 \end{aligned}$$

Gradient Vector Module

$$|G| = \sqrt{G_t^2 + G_u^2 + G_v^2} = \sqrt{(-0,3661)^2 + (-0,1417)^2 + (-1)^2} = 1,0743.$$

The derivative of the scalar field (10) in the direction of (11)

$$\begin{aligned}
 \frac{dF}{dl} &= \frac{G_t}{|G|} \cos \alpha + \frac{G_u}{|G|} \cos \beta + \frac{G_v}{|G|} \cos \gamma = \\
 &= -\frac{0,3661}{1,0743} 0,6 - \frac{0,1417}{1,0743} 0 - \frac{1}{1,0743} 0,8 = -0,9492
 \end{aligned}$$

To obtain images of level surfaces, the right side of equation (12) must be substituted in (1). Then use the parametric equations of the family of level surfaces to visualize, jointly or separately, representatives of this family using computer graphics tools.

Figure 2 shows an axonometric projection of the level surfaces (12) at $i = -1, 0, 1$, $\Delta C = 0,5$. In Figure 3-5, the level surfaces are shown separately from each other.

The method of reading the intrinsic equation (12) is in the mental synthesis of the shape of the level surface when analyzing the shape of its coordinate lines. The coordinate line $t = \text{const} = p$, according to equation (12), is a sinusoid $v = h \sin u \cos np$, whose abscissa axis is deformed into a circle centered at the point $x = R + r \cos p$, $y = R + r \sin p$, $z = 0$ and with radius $(r + i\Delta C)$. The amplitude of this deformed sinusoid is $h \cos np$. The ordinate v is deposited on the rays in the semi-plane $t = p$ from the deformed axis of abscissae.

The coordinate line $u = \text{const} = q$ is located on the surface of a cone of revolution normal to the determinant torus whose base is the circle $u = q$. The coordinate line $u = \text{const}$ is the cosine of $v = h \sin q \cos nt$, whose abscissa axis is deformed into a circle of radius $[R + (r + i\Delta C) \cos q]$ with the centre at the point $[0, 0, (r + i\Delta C) \sin q]$.

The amplitude of the cosine curve is $h \sin q$, the ordinate v is deposited on the generators of the normal cone from the deformed abscissa. Since $n = 6$, the cosine in the interval $0 \leq u \leq 2\pi$ has six periods. It should be emphasized that the amplitudes of both coordinate lines are variable: the amplitude of the sinusoid $t = \text{const}$ depends on t , the cosine-wave amplitude $u = \text{const}$ - on u .

The influence of the values of the parameters of the functions (12) and (1) on the shape of the level surface can be traced from the images of these surfaces with the same internal equation (12), giving different values to the parameters of both function (12) and functions (1).

Fig. 2. Surfaces of the level $v = h \cdot \sin u \cdot \cos(n \cdot t) + i \cdot \Delta C$ of a scalar field $F = h \cdot \sin u \cdot \cos(n \cdot t) - v$ with parameters: $R=5$, $r=2$, $h=0,75$, $n=6$, $u=-1,8...3,1$, $\Delta C=0,5$ $t=0...2\pi$.

Fig. 3. Surfaces of the level $i=-1$

For each set of parameters, an image of three level surfaces is presented. Thus, the value of the parameter of the family is influenced on the shape of the level surface $i\Delta C$.

The parameter n , geometrically determines the number of waves on the surface in the interval $0 \leq t \leq 2\pi$.

The parameter h can be interpreted as the coefficient of amplitudes. In Figures 2-5, $h = 0.75$.

The parameters R and r of the torus-determinant of the system of normal toroidal coordinates are direct parameters of the shape of the level surfaces.

Fig. 4. Surfaces of the level $i=0$

Fig. 5. Surfaces of the level $i=1$

The shape of the level surface, more precisely of its unit, can be varied by changing the intervals of values of the normal toroidal coordinates entering the right-hand side of the internal equation (12).

Figures 2-5 are obtained for different intervals of u coordinate values (these intervals are indicated in the figures). They are not parameters of the shape of the surface as a whole. More likely, they can be defined as the shape of the unit.

3 Results

1. Expressions of the differential characteristics of scalar and vector fields in normal toroidal coordinates are given.
2. The technique of "reading" the intrinsic equations of the level surfaces of scalar fields, represented in normal toroidal coordinates, is shown.

4 Discussion

1. For maximum clarity in the study of fields that are presented in normal toroidal coordinates, there is the possibility of computer visualization of the level surfaces of fields of complex structure.
2. In the example presented, the mathematical apparatus for describing the surfaces of the level of scalar fields is given. The possibilities of the influence of the values of the parameters and the intervals of the change of variables on the shape of the level surfaces are also shown.
3. In accordance with the above expressions, it is possible to solve various problems:
 - calculating the parameters of the scalar field at a given point for known torus-spacer parameters and point coordinates, with subsequent visualization of the field structure;
 - determination of the value of C at which the level surface of the field represented in normal toroidal coordinates passes through a point with given coordinates;
 - Computer visualization of fields represented in normal toroidal coordinates for different values of R and r ;
 - calculation of the parameters of the vector field (divergence, Laplacian, rotor) in normal toroidal coordinates.

5 Conclusion

The representation of field theory in special coordinates dramatically simplifies the calculation of field characteristics when the field source has a complex structure. Further research on this subject will concern representation of field characteristics in conic coordinates.

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