Structure of laminar flows with oppositely-rotating coaxial layers

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Abstract. The article is devoted to the theoretical study of laminar flows with the coaxial layers rotating in opposite directions moving along the pipe. These flows have a wide practical application potential in technologies of mixing multiphase and heterogeneous media in microbiology, chemistry, ecology, heat engineering, power engineering, civil engineering and engine and rocket science. Such flows have a complicated three-dimensional structure. The theoretical model of the test flow is based on the Navier – Stokes's equations and Fourier – Bessel's method of expansion of differential equations. The article presents the formulas and graphs showing the radial-axial distributions of tangential, axial and radial flow velocities, stream functions and viscous vortex components. The authors made the theoretical analysis of the kinematic structure of such flows.

1 Introduction

Fluid flows with oppositely-rotation coaxial layers have a complicated three-dimensional structure (**Fig. 1**). Such currents arise in cyclones and behind Francis turbines [1-5].

Fig. 1. The structure of the flow with cocurrent counter-rotation coaxial layers in a cylindrical pipe: **a** – classic profiles of the azimuthal (u_θ) and axial (u_x) velocities of two-layer flow; **b** – components of velocity vectors, of the vortex of an elementary particle of a fluid and of viscous stress tensor.

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The beginning of their studies has been associated with the development of effective methods dissipation of water flow kinetic energy in the deep-set spillways [6]. In the process of these studies in the Russian-language technical and normative literature, the term “contrvihrevoe techenie” (“counter-vortex flow”) have formed¹. Later, the range of application of such flows expanded [7], since the opposite rotation of interacting layers in the turbulent range allows one to obtain a number of effects, among which one should note the intensive mixing of the medium. This effect has a wide practical application in technologies including mixing of multiphase and heterogeneous media in microbiology, chemistry, ecology, heat, power and civil engineering, engine and rocket engineering. Theoretical study of laminar counter-vortex flows is interesting in that it allows for identifying general physical laws of their hydrodynamics. Development of these technologies requires knowledge of hydrodynamics of counter-vortex flows, which earlier have been studied by physical [8] and numerical [9, 10] simulations. The purpose of our research is improvement and analysis of the theoretical model of laminar flow with oppositely-rotation coaxial layers.

2 Method of research

In the article presents the results of theoretical studies of the structure of counter-vortex flows based on recent publications [10-15]. In these publications in the cylindrical coordinate system $r - \theta - x$ (**Fig. 1**), the steady ($\partial/\partial t = 0$, where t is the current time) symmetric relative to the pipe axis ($\partial/\partial \theta = 0$) flow of incompressible fluid in the laminar range is described by the Navier – Stokes’s equations [16]

$$u_r \frac{\partial u_r}{\partial r} + u_x \frac{\partial u_r}{\partial x} - \frac{u_\theta^2}{r} = -\frac{\partial}{\partial r} \left(\Pi + \frac{P}{\rho} \right) + \nu \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial u_r}{r \partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial x^2} \right), \quad (1,a)$$

$$u_r \frac{\partial (ru_\theta)}{r \partial r} + u_x \frac{\partial u_\theta}{\partial x} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial u_\theta}{r \partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial x^2} \right), \quad (1,b)$$

$$u_r \frac{\partial u_x}{\partial r} + u_x \frac{\partial u_x}{\partial x} = -\frac{\partial}{\partial x} \left(\Pi + \frac{P}{\rho} \right) + \nu \left(\frac{\partial^2 u_x}{\partial r^2} + \frac{\partial u_x}{r \partial r} + \frac{\partial^2 u_x}{\partial x^2} \right), \quad (1,c)$$

where u_r , u_θ , u_x are radial, azimuthal and axial velocity components; P , Π are pressure and potential of external mass forces; ρ and ν are density and kinematic viscosity of fluid.

3 Results and Discussion

3.1 Distributions of flow velocities

Assuming that the radial velocity is much smaller than the azimuthal ($u_r \ll u_\theta$) and axial ($u_r \ll u_x$) ones and taking classical Ozeen approximation [17] equations (1) are scaled to normalized by pipe radius R and average flow velocity $V = Q/\pi R^2$ (where Q is fluid flow rate) closed system of two linear differential equations of parabolic type with two unknowns normalized velocities u_θ and u_x [13]

$$\frac{\partial u_\theta}{\partial x} = \frac{1}{\text{Re}} \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial u_\theta}{r \partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial x^2} \right), \quad (2,a)$$

$$\frac{\partial}{\partial x} \left(\frac{u_\theta^2}{r} \right) + \frac{\partial^2 u_x}{\partial x \partial r} = \frac{1}{\text{Re}} \cdot \frac{\partial}{\partial r} \left(\frac{\partial^2 u_x}{\partial r^2} + \frac{\partial u_x}{r \partial r} + \frac{\partial^2 u_x}{\partial x^2} \right), \quad (2,b)$$

where Re is Reynolds number, $\text{Re} = VR/\nu$.

The boundary conditions for equations (2) on the walls of the cylindrical pipe ($r=1$), its axis ($r=0$), at the inlet ($x=0$) to the zone of interaction of oppositely rotating coaxial layers (the active zone) and at an infinite distance from the entrance ($x=\infty$) are of the form

$$u_\theta = u_x = 0 \quad \text{for } r = \pm 1 \quad \text{and } x > 0, \quad (3,a)$$

$$u_\theta = \frac{\partial u_x}{\partial r} = 0 \quad \text{for } r = 0 \quad \text{and } x > 0, \quad (3,b)$$

$$u_\theta = \Omega_0 r + \frac{\Gamma_0}{r} + A_0 J_1(\mu r) \quad \text{for } x = 0 \quad \text{and } -1 \leq r \leq 1, \quad (3,c)$$

$$u_\theta = 0; \quad u_x = 2(1-r^2) \quad \text{for } x = \infty \quad \text{and } -1 \leq r \leq 1, \quad (3,d)$$

where Ω_0 , Γ_0 and A_0 are pre-assigned normalized coefficients of radial distribution of the azimuthal velocity at the entry to the active zone; $J_1(\dots)$ is the Bessel function of the first kind of the first order; μ is constant not equal to λ_n ; λ_n is Bessel function root $J_1(\lambda_n) = 0$.

Decisions of the system of equations (2) with the boundary conditions (3) are given in the above publications [10-15] which in chronological order improve and complement each other. But in general, the most comprehensive solutions for today are the radial-axial distributions of azimuthal and axial velocities given in monograph [10] in the form of Fourier – Bessel's series or the products of Fourier – Bessel's series

$$u_\theta(r, x, \text{Re}) = 2 \sum_{n=1}^{\infty} G_n \frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right), \quad (4)$$

$$\begin{aligned} u_x(r, x, \text{Re}) = & 2(1-r^2) - \sum_{i=1}^{\infty} \frac{4}{\lambda_i^2} \left[1 - \frac{J_0(\lambda_i r)}{J_0(\lambda_i)} \right] \exp\left(-\lambda_i^2 \frac{x}{\text{Re}}\right) + \\ & + 2 \left\{ \sum_{n=1}^{\infty} \frac{G_n}{\lambda_n^2} \left[1 + \frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2} r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \right. \\ & - \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \frac{J_0(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\ & \left. - \sum_{n=1}^{\infty} \frac{G_n^2}{\lambda_n^2} \left[\frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2} r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-2\lambda_n^2 \frac{x}{\text{Re}}\right) \right\}, \quad (5) \end{aligned}$$

where $J_0(\dots)$ and $J_2(\dots)$ are Bessel functions of the first kind of the zero and second orders; λ_i are real zeros of the Bessel function of the first kind of the second order $J_2(\lambda_i) = 0$; λ_n and λ_k are real zeros of the Bessel function of the first kind of the first order $J_1(\lambda_n) = 0$, $J_1(\lambda_k) = 0$; G_n and G_k are constants n and k partial solutions

$$G_n = \Gamma_0 \left[\frac{1}{J_0(\lambda_n)} - 1 \right] - \Omega_0 - \frac{A_0 J_1(\mu)}{1 - (\mu/\lambda_n)^2}, \quad G_k = \Gamma_0 \left[\frac{1}{J_0(\lambda_k)} - 1 \right] - \Omega_0 - \frac{A_0 J_1(\mu)}{1 - (\mu/\lambda_k)^2}. \quad (6)$$

When finding the distributions (4) and (5) in [10] the second partial derivatives in the right-hand sides of equations (2) $\partial^2 u_\theta / \partial x^2$ и $\partial^2 u_x / \partial x^2$ were negligibly small, and the condition was laid for conservation of the volume flow integral

$$\int_0^1 u_x r dr = 1 \quad \text{for } x \geq 0.$$

In the same publication [10] from the continuity equation

$$\frac{\partial(ru_r)}{r\partial r} + \frac{\partial u_x}{\partial x} = 0$$

the radial-axial distribution of radial velocities has been also derived

$$\begin{aligned} u_r(r, x, \text{Re}) = & -\frac{2}{\text{Re}} \sum_{i=1}^{\infty} \left[r - \frac{2J_1(\lambda_i r)}{\lambda_i J_0(\lambda_i)} \right] \exp\left(-\lambda_i^2 \frac{x}{\text{Re}}\right) + \\ & + \frac{1}{\text{Re}} \left\{ \sum_{n=1}^{\infty} G_n \left[r + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n r \sqrt{2})}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\ & + \sum_{n=1}^{\infty} \frac{G_n}{\lambda_n^2} \left[r + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n r \sqrt{2})}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \lambda_k^2 \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\ & - 2 \sum_{n=1}^{\infty} \frac{G_n \lambda_n J_1(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \left[\sum_{k=1}^{n-1} \frac{G_k J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\ & + \sum_{k=n+1}^{\infty} \frac{G_k J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \left. \right] - 2 \sum_{n=1}^{\infty} \frac{G_n J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \times \\ & \times \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k^2 J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k^2 J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \right] + \\ & + 2 \sum_{n=1}^{\infty} \frac{G_n J_0(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\ & + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \left. \right] + 2 \sum_{n=1}^{\infty} \frac{G_n J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \times \\ & \times \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k^3 J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k^3 J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \right] - \\ & - 2 \sum_{n=1}^{\infty} G_n^2 \left[\frac{rJ_0^2(\lambda_n r) + rJ_1^2(\lambda_n r)}{J_0^2(\lambda_n)} + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n r \sqrt{2})}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp\left(-2\lambda_n^2 \frac{x}{\text{Re}}\right) \Big\}. \quad (7) \end{aligned}$$

Fig. 2 shows the radial-axial distribution of azimuthal, axial and radial velocities in laminar counter-vortex currents. The calculations of the graphs were made using theoretical formulas (4) - (7) with Reynolds numbers equal to $\text{Re} = 500$. For comparison purposes, two flows are defined at the entrance to the active zone: the first (mode 1) two-layered with the parameters $\Omega_0 = 4.241$, $\Gamma_0 = -1.216$, $A_0 = 0$; the second (mode 2) is a four-layered one with the parameters $\Gamma_0 = \Omega_0 = 0$, $A_0 J_1(\mu) = 0.1584$, $\mu = 13.3$. The distances from the entry section of the active zone ($x = 0$) to the cross-sections, in which the velocity profiles were calculated, are shown in **Fig. 2** next to the calculated profiles in fractions of the pipe radius ($x = 5R, 10, 20, 40, 80$).

The analysis of the profiles of azimuthal velocities shows that the counter-vortex flows within the active zone are transformed into longitudinal-axial currents without swirling. The length of the active zone - the zone of intensive viscous vortex suppression of interacting layers with laminar two-layered flow is equal to 40 pipe radii. With four-layered flow the length of the active zone is reduced twice - to 20 pipe radii. Viscosity of the medium has a significant impact on the length of the active zone. Analyzing the distribution function (4), where viscosity is present in the Reynolds number, it is easy to establish that

Fig. 2. Distributions of azimuthal, axial and radial velocities in flows with oppositely swirled coaxial layers: above – mode 1, below – mode 2.

its change affects in inverse proportion the extent of the active zone: with increasing viscosity the length of the active zone proportionally reduces, with decreasing thereof the length of the active zone proportionally increases.

Concerning the distribution of axial velocities, it should be noted that common for the two counter-vortex flows presented is a strong return flow with significant negative velocities observed in the axial area at the beginning of the active zone. Further lengthwise the pipe, return currents decrease and disappear, transforming into velocity deficiency characteristic for the flow behind a poorly-streamlined body. Similar velocity deficiency is observed in the zones between oppositely rotating coaxial layers. Outside the return current in the near-axial flow the velocities in the stratum along the current can significantly exceed the average velocity ($V = 1$). Calculations show that the mass balance at the beginning of the active zone is maintained at the expense of this phenomenon. Later, in the process of reforming the profile of axial velocities, the layers that are increasingly distant from the zone of return currents are involved. Further along the length of the pipe, the displacement of the zone of maximum axial velocities is directed from the periphery to its center and, in the degeneration of the circulation, the axial velocities reach their maxima on the axis. The parabolic Poiseuille profile forms to the site at a distance of 80 radii from the entrance to the active zone. Since the axial velocity profiles form in the counter-vortex currents to a great extent due to swirling the current, the longitudinal flow rate within the active zone has the properties of the secondary flow dependent on the azimuthal component of the velocity.

In **Fig. 2** it can be seen that the values of the radial velocities are one or two orders lower than the azimuthal and axial values. And the radial velocities will be the lower, the higher is the Reynolds number, which stands in the denominator of the obtained distribution (7). It can be said that the assumption made above on the basis of the previously obtained experimental data [6, 8, 10] that it is possible to neglect the radial component u_r in the equations of the dynamics of a viscous fluid (but not in the continuity equation, since in this case the mass balance is violated) finds analytical confirmation. It can further be seen that in the counter-vortex flows on the radial velocity profiles we observe flows directed exclusively toward the axis of the pipe (radial velocities on the profiles are negative). But this will not be observed in all the sections. It was noted above that at the beginning of the active zone in the area of the pipe we see a strong return flow with high negative axial velocities, both in the two-layer and four-layer counter-vortex flows. Such currents inevitably form around the areas with recirculation motion, while divergent currents of fluid from the axis to the walls of the pipe will be observed in the sections directly at the beginning of the active zone. What we observe on the obtained profiles of radial velocities can be characterized as an on-going transformation of the weakening counter-vortex flows to Poiseuille's flow.

3.2 Distribution of currents function

In the conditions of laminar flow the formation of the recirculation area in the zone of the flow axis excludes mass transfer between this area and transit flow streamlining it. Thus, the liquid in the recirculating "bubble" does not flow downstream. In accordance with the property of the streamlines, the tangents to which at any point coincide with the direction of the velocity vectors of elementary particles of the liquid located on them at a given instant of time, and in steady flows coincide with the trajectories of the particles, it can be assumed that the described recirculation area is a ring toroidal θ -vortex with closed elliptical lines of current Ψ and is limited by an isoline with zero value ($\Psi = 0$) and a deceleration point on

the pipe axis. With the aim of identifying recirculating areas in the streams under study we find the distribution of currents function Ψ , whose value on the pipe axis at $r = 0$ is equal $\Psi = 0$ for all $x \geq 0$. The radial-longitudinal distribution of Ψ can be found through a definite integral

$$\Psi(r, x, \text{Re}) = \int_0^{\Psi} d\Psi = \int_0^r u_x r dr.$$

Integrating with due regard of the above distribution u_x , we find

$$\begin{aligned} \Psi(r, x, \text{Re}) = & r^2 \left(1 - \frac{r^2}{2} \right) - 2r \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \left[r - \frac{2J_1(\lambda_i r)}{\lambda_i J_0(\lambda_i)} \right] \exp \left(-\lambda_i^2 \frac{x}{\text{Re}} \right) + \\ & + r \left\{ \sum_{n=1}^{\infty} \frac{G_n}{\lambda_n^2} \left[r + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n \sqrt{2}r)}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp \left(-\lambda_n^2 \frac{x}{\text{Re}} \right) \sum_{k=1}^{\infty} G_k \exp \left(-\lambda_k^2 \frac{x}{\text{Re}} \right) - \right. \\ & - 2 \sum_{n=1}^{\infty} \frac{G_n J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp \left(-\lambda_n^2 \frac{x}{\text{Re}} \right) \left[\sum_{k=1}^{n-1} \frac{G_k J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp \left(-\lambda_k^2 \frac{x}{\text{Re}} \right) + \right. \\ & + \sum_{k=n+1}^{\infty} \frac{G_k J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp \left(-\lambda_k^2 \frac{x}{\text{Re}} \right) \left. \right] + 2 \sum_{n=1}^{\infty} \frac{G_n J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp \left(-\lambda_n^2 \frac{x}{\text{Re}} \right) \times \\ & \times \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp \left(-\lambda_k^2 \frac{x}{\text{Re}} \right) + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp \left(-\lambda_k^2 \frac{x}{\text{Re}} \right) \right] - \\ & - \sum_{n=1}^{\infty} \frac{G_n^2}{\lambda_n^2} \left[\frac{rJ_0^2(\lambda_n r) + rJ_1^2(\lambda_n r)}{J_0^2(\lambda_n)} + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n \sqrt{2}r)}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp \left(-2\lambda_n^2 \frac{x}{\text{Re}} \right) \Big\}. \end{aligned}$$

The calculated current line fields of the counter-vortex flows under study with $\text{Re} = 500$ are shown in Fig. 3 at the mixing chamber section of a length of twenty its radii. This length is sufficient for visualization of all areas with recirculation motion in the zone of counter-vortex flow.

Calculations show the presence of several areas with recurrent circulation in the counter-vortex flows. Moreover, the axial recirculation areas, which are toroidal θ -vortices, both in the two-layer (Fig. 3,a) and in the four-layer (Fig. 3,b) counter-vortex flows are highly developed in the axial and radial directions. In the case of a two-layer counter-vortex flow the area of recirculation motion is substantially more stretched along the pipe with the deceleration point at a distance of $15.5R$ from the beginning of the active zone than in the four-layer flow, where the deceleration point is located at a distance of $8R$. However, the four-layer flow is much more developed in the radial direction than the two-layer flow. In a two-layer counter-vortex flow one more area with a recirculation motion can be seen in the immediate vicinity of the inlet section of the active zone. Such a cellular structure is characteristic for the areas of decay of circulation-longitudinal flows, for example, in the form of a so-called bubble form [10]. In comparison with the two-layered four-layer counter-vortex flow has an even more complicated cellular structure, which contributes to its even more intense decay.

3.3 Vortex structure of the flow

Let's consider the last kinematic characteristic - the vortex structure of the counter-vortex flows. Under the assumptions made, using the distributions (4) and (5), for the components of the vortex structure of currents we find

Fig. 3. Current lines in two-layer (a) and four-layer (b) counter-vortex flows.

$$\begin{aligned}
 \text{rot}_r U &= -\frac{\partial u_\theta}{\partial x} = \frac{2}{\text{Re}} \sum_{n=1}^{\infty} G_n \frac{\lambda_n J_1(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right), \\
 \text{rot}_\theta U &= -\frac{\partial u_x}{\partial r} = 4r + 4 \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r)}{\lambda_i J_0(\lambda_i)} \exp\left(-\lambda_i^2 \frac{x}{\text{Re}}\right) - \\
 &\quad - 2 \left[\sum_{n=1}^{\infty} G_n \frac{\sqrt{2} J_1(\lambda_n \sqrt{2} r)}{\lambda_n J_2(\lambda_n \sqrt{2})} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\
 &\quad + \sum_{n=1}^{\infty} G_n \frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \frac{J_0(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \\
 &\quad + \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \frac{\lambda_k J_1(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\
 &\quad \left. - \sum_{n=1}^{\infty} G_n^2 \frac{\sqrt{2} J_1(\lambda_n \sqrt{2} r)}{\lambda_n J_2(\lambda_n \sqrt{2})} \exp\left(-2\lambda_n^2 \frac{x}{\text{Re}}\right) \right], \\
 \text{rot}_x U &= \frac{\partial(r u_\theta)}{r \partial r} = 2 \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right).
 \end{aligned}$$

The isometric images of the vortex fields in the active zone of the investigated counter-current flows are shown in Fig. 4 and 5. Since flows symmetrical with respect to the axis of the pipe in the drawings, the radial range is taken from 0 to $r/R = 1$. The calculations are performed at Reynolds numbers equal to $\text{Re} = 500$. In the longitudinal direction, the calculated area includes the entire active zone of a length of up to 40 radii of the pipe, on which there are counter-vortex flows. In order to represent the vortex structure of the counter-vortex flows, one should turn to Fig. 1, *b*, where the components of the vortex vector of the isolated elementary volume of liquid are shown. It should be borne in mind that the positive value of the vortex corresponds to the rotation of the elementary particle counterclockwise, and the negative one – clockwise.

The analysis shows that cascades of concentric vortices of different signs are generated in the inner layers of the counter-vortex flows in the active zone along the radius. There is no similar cellular structure of the vortex fields in either axial flows, where the generation of vortices of one sign is due to the viscous inhibition of the liquid in the wall layer, or in circularly longitudinal layers with unidirectional swirling of layers, in which there are two zones of generation of vortices of the opposite sign: viscous inhibition and an axial vortex zone in the active zone of the current. Thus, the structure of counter-vortex flows is formed under the prevailing influence of internal processes in the zones of generation of cascade vortex fields due to the forces of viscosity. In general, this is the cardinal difference between the counter-vortex currents from axial and circular longitudinal ones.

The cellular cascade structure of the vortex fields in the counter-vortex flow promotes their intensive decay and the mutual viscous diffusion (dissipation) of the circulation of the interacting oppositely swirled layers. It should be noted that with the number of layers of the counter-vortex flow, the multiplicity of the cellular structure of the vortex fields is proportionally increasing. Nascent at the entrance to the active zone, the vortex cascades are then rapidly accelerated by viscosity forces. Actually, this is the physical cause of the dissipation of the mechanical energy of the flow and its transformation into heat. However, these processes will be slower, and cascade vortices generated at the entrance to the active zone will penetrate to more remote areas of the flow less weakened, the greater is the Reynolds number (the lower the viscosity of the liquid) and the smaller is the number of interacting layers.

Fig. 4. Vortex fields in two-layers counter-vortex flow.

Fig. 5. Vortex fields in four-layers counter-vortex flow.

Turning to the quantitative results, we note that the calculations showed that with Reynolds number of $Re = 500$ and the given parameters of the counter-vortex flows at the entrance to the active zone the azimuthal ($rot_\theta U$) and axial ($rot_x U$) vortices reach the values of the order of 400 standardized units (Fig. 4, *b,c* and 5, *b,c*). This is several times higher than the maximum values of the vortex fields generated at the same Reynolds numbers in circulation-longitudinal flows with any degree of flow swirling and, especially, in axial Poiseuille flow, where the only non-zero component of the vortex $rot_\theta U$ has the maximum value at the pipe wall equal to 4 standardized units

$$rot_\theta U = -\frac{\partial u_x}{\partial r} = -\frac{\partial}{\partial r} \left[2(1-r^2) \right] = 4r|_{r=1} = 4,$$

i.e. is two orders lower. Radial vortices ($rot_r U$) reach values 5 – 8 standardized units (Fig. 4, *a* and 5, *a*). It reflects the fact that in counter-vortex flows longitudinal partial derivatives of velocities are about two orders lower than radial ones.

In conclusion, we note an important feature of viscous laminar counter-vortex flow associated with the fact that all their structural characteristics described by distributions in the form of Fourier – Bessel's series

$$f(r, z, Re) = Re^b \sum_{n=1}^{\infty} C_n J_m(\lambda_n r) \exp\left(-\lambda_n^2 \frac{x}{Re}\right)$$

either in the form of products of the Fourier – Bessel's series

$$f(r, z, Re) = Re^b \sum_{n=1}^{\infty} C_n J_m(\lambda_n r) \exp\left(-\lambda_n^2 \frac{x}{Re}\right) \sum_{k=1}^{\infty} C_k J_m(\lambda_k r) \exp\left(-\lambda_k^2 \frac{x}{Re}\right),$$

where b is the exponent ($b = 0, -1$); λ_n and λ_k are real zeros of the Bessel function of the first kind of m order ($J_m(\lambda_n) = 0, J_m(\lambda_k) = 0$); C_n, C_k are constants of integration.

Consequently, for flows with equal constants C_n and C_k at equal relative distance from the inlet cross-section (x/Re), the radial profiles of an arbitrary flow characteristics will be the same for $b = 0$, i.e., if the Reynolds numbers are not taken as the sign of the sum of the series; In the other case (for $b = -1$), the shape of the profiles of the compared flows will remain unchanged, but their relative scale will be determined by the Reynolds number ratio. An increase in the Reynolds number at $b = 0$ leads to a “stretching” of the process along the axial coordinate, i.e. profile “drifting” downstream by a proportional distance; and for $b = -1$, the “drift” of the profile is accompanied by a proportional decrease in its scale.

The study of the flow structure also includes the description of the viscous stress tensor and local stability zones. These characteristics will be considered in the next article devoted to the hydrodynamic structure of counter-vortex flows.

Conclusions

1. Liquids flows in pipes with cocurrent counter-rotation coaxial layers have a complicated three-dimensional structure. Earlier, the study of such currents was performed by methods of physical and numerical simulation. In the article authors makes an attempt to study and analyze counter-vortex flows kinematic structure using the methods of theoretical hydromechanics.

2. The theoretical model of the investigated flow is based on the Navier – Stokes's equations and the Fourier method of expansion of the differential equations. As a result, the article presents the formulas of radial-axial distributions of tangential, axial and radial flow velocities, as well as current functions and components of viscous vortices, representing series or products of series of Bessel's functions.

3. Analysis of the distribution of tangential velocities shows that counter-vortex flows

within the active zone (the zone of intensive viscous suppression of interacting oppositely rotating coaxial swirling layers) are transformed into longitudinal-axial flows without swirling. The length of the active zone for a laminar two-layered flow is equal to 40 pipe radii, with a four-layered flow the length of the active zone is reduced twice – to 20 radii. The change in the viscosity of the liquid inversely proportionally affect to the length of the active zone: with increasing viscosity the length of the active zone proportionally reduces, with decreasing - proportionally increases.

4. For distributions of axial velocities in counter-vortex flows are characterized by formation at axis region at the beginning of the active zone a return current with significant negative velocities. Further along the length of the pipe the return current decreases and disappear, transforming into velocity deficiency characteristic for the flow behind a poorly-streamlined body. The parabolic Poiseuille profile forms at a distance of 80 radii from the entrance to the active zone.

5. The radial velocities are one or two orders lower than the azimuthal and axial ones. The radial velocities will be lower, then lower the viscosity of the liquid.

6. In the conditions of laminar flow the formation in the zone of the axis of the return current leads to the generation a recirculation region, the mass exchange between this region and transit flow is absent. Thus, the liquid inside the recirculating “bubble” does not flow downstream. The near-axis recirculation areas are toroidal θ -vortices, in the two-layer and four-layer counter-vortex flows are highly developed in the axial and radial directions with braking point on the pipe axis. With a two-layer counter-vortex flow the deceleration point is located at a distance of $15.5R$ from the beginning of the active zone, in the four-layer - $8R$.

7. The analysis shows that in the counter-vortex flows in the active zone along the radius cascades of concentric vortices of different signs are generated. There is no similar cellular structure and intensity of the vortex fields in either axial flows or in circulation-longitudinal flows, in which the maximum intensity of the vortices is two orders of magnitude lower. The cellular cascade structure of the vortex fields in the counter-vortex flows promotes their intensive decay and the mutual viscous diffusion of the circulation of the interacting oppositely swirled layers. With the number of layers of the counter-vortex flow, the multiplicity of the cellular structure of the vortex fields is proportionally increasing. Nascent at the input of the active zone, the vortex cascades are quickly suppressed by the forces of viscosity, which is the physical cause of the effective dissipation of the mechanical energy of the flow. These processes will be slower, and cascade vortices generated at the entrance to the active zone will penetrate to more remote areas of the flow less weakened, the lower the viscosity of the liquid and smaller number of interacting layers.

8. Since all the structural characteristics of counter-vortex flows are described by distributions in the form of series or products of Fourier-Bessel's series, which include the Reynolds number in the exponent $\exp(-\lambda^2 x / \text{Re})$ and then multiplied by Re^b , where $b = 0$ or $b = -1$, then increase in the Reynolds number at $b = 0$ leads to a “stretching” of the process along the axial coordinate, i.e. profile is “drift” downstream by a proportional distance; and for $b = -1$, the “drift” of the profile is accompanied by a proportional decrease in its scale.

9. In further studies, the authors consider it necessary to perform an analysis of the hydrodynamic structure of counter-vortex flows, including a description of the viscous stress tensor and local stability zones.

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