An evaluation on the performance of two simple triangular bending plate elements

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Abstract. This paper will study and compare two different three-node triangular bending plate elements with three degree of freedom per node, i.e. MITC3 and DKMT. Both elements, which were developed based on Reissner-Mindlin plate theory and independent shear strain field, have simple formulation and have already been used widely. In this paper, numerical tests for circular plate case are conducted to verify the performance and show the convergence of these two triangular elements.

1 Introduction

In structural modelling, the use of triangular element is interesting due to its simplicity and flexibility. Three-node triangular element is mostly used for complex configuration. However, research on triangular is not as much as quadrilateral element. In finite element analysis, many researches show that quadrilateral element have a good performance. As a result, many analyses prefer to use the quadrilateral element [1-10]. This condition should encourage researchers to develop study about triangular element.

The challenge in finite element is how to generate a simple and applicable element formulation to reduce the computational cost, yet still has high accuracy and good convergence.

MITC (Mixed Interpolation of Tensorial Components) concept has been applied for triangular element. MITC3, proposed by Lee and Bathe [11], is one of the popular bending elements. There are quite a lot of studies and developments of MITC3 [12-15]. This three-node triangular element has a simple and general formulation. The behaviour of the element is isotropic, meaning that the stiffness matrix of the element does not depend on the sequence of node numbering.

Previously, Katili [16] proposed simple formulation for triangular bending plate element. The element, called DKMT, is an extension of DKT (Discrete Kirchhoff Triangular) element [17] which is a simple, efficient and reliable element for analysis of thin plates. In the case of thin plate problem, the solution obtained with DKMT element converged to that given by DKT element.

The purpose of this paper is to compare formulation and performance of two triangular elements, i.e. MITC3 and DKMT. Both of them have simple formulation. At the end of this paper, we conduct numerical studies for circular bending plate to compare and evaluate the convergence of both elements.

2 Formulation of DKMT

The formulation of DKMT plate element is based on Reissner-Mindlin hypothesis, which only require C0 continuity, and generated by assumed natural strain concept. Transversal shear deformation for this element is expressed with special interpolation called shear projection method. The triangular element has three-degrees of freedom per node (Fig.1).

![Triangle element and the degrees of freedom](image)

Fig. 1. Triangle element and the degrees of freedom

2.1 Bending strain matrix

The DKMT element was first published in 1993 [16]. DKMT element has 3 nodes with 3 degrees of freedom each, which are: \( w \) (translation in the \( z \) direction), \( \beta_x \) (rotation in the \( z-x \) plane) and \( \beta_y \) (rotation in the \( z-y \) plane). Incomplete quadratic rotation fields for \( \beta_x \) and \( \beta_y \) are considered in terms of rotations at the three corners and a temporary variable at mid-side \( i-j \) (Fig.2).

The displacement function is given as:

\[
\begin{align*}
\mathbf{w} &= \sum_{i=1}^{3} N_i w_i \\
\mathbf{\beta} &= \sum_{i=1}^{3} N_i \mathbf{\beta}_i + \sum_{k=1}^{6} \mathbf{B}_k \mathbf{\Delta} \mathbf{\beta}_{ik}
\end{align*}
\]  

(1)

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where \( N_i \) are the linear shape functions and \( P_k \) are the quadratic functions:
\[
N_1 = \lambda ; \quad N_2 = \xi ; \quad N_3 = \eta ; \quad \lambda = 1 - \xi - \eta
\]
\[
P_k = 4\lambda \xi ; \quad P_3 = 4\beta \eta ; \quad P_0 = 4\lambda \eta
\]
\( C_k \) and \( S_k \) are the direction cosines of side \( i-j \) (Fig. 2).

\[\{\psi\} = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} = \begin{bmatrix} \beta_{x,i} \\ \beta_{y,i} \end{bmatrix} + \begin{bmatrix} \beta_{x,i} \beta_{y,i} \beta_{x,i} \beta_{y,i} \end{bmatrix}
\]

where \( \beta_{x,i} \) and \( \beta_{y,i} \) denote the first derivatives of \( \beta \) with respect to \( x \) and \( y \), respectively.

The relation between curvature and nodal displacement is expressed in the equation below:
\[
\{\chi\} = \begin{bmatrix} \chi_x \\ \chi_y \end{bmatrix} = \begin{bmatrix} \beta_{x,i} \\ \beta_{y,i} \end{bmatrix} + \begin{bmatrix} \beta_{x,i} \beta_{y,i} \beta_{x,i} \beta_{y,i} \end{bmatrix}
\]

where:
\[
\{u_n\} = \{u_n\}^T = \ldots w_j \beta_{x,i} \beta_{y,i} \ldots
\]

\[
\{\Delta \beta_{n,i}\} = \{\Delta \beta_{n,i}\}^T = \{\Delta \beta_{n,i} \Delta \beta_{n,i} \Delta \beta_{n,i}\}^T
\]

From eq. (1.4), we obtain the expression of bending strain matrix,
\[
[B_b] = \begin{bmatrix} 0 & N_{i,x} & 0 \\ \ldots & 0 & 0 & N_{i,y} & \ldots \\ 0 & 0 & N_{i,x} & \ldots \\ \ldots \end{bmatrix}_{i=1,2,3}
\]
\[
[B_{b,0}] = \begin{bmatrix} P_{i,x} C_k \\ P_{i,x} S_k \\ S_{i,y} C_k \\ S_{i,y} S_k \end{bmatrix}_{i=4,5,6}
\]

\( N_{i,x} \), \( P_{i,x} \), \( N_{i,y} \), and \( P_{i,y} \) denote the first derivatives of \( N_i \) and \( P_i \) with respect to \( x \) and \( y \), respectively.

### 2.2 Shear strain interpolation

Shear deformation field is approximated linear in each element.
\[
\{\gamma\} = \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix} = \sum_{i} N_i \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix}
\]

where \( \sum\gamma_x \) and \( \sum\gamma_y \) are the shear deformations at nodes-i.

Shear deformation is assumed constant along the edge of element (Fig.3). Shear deformation at node-i is obtained from projection of constant shear deformation \( \sum\gamma_x \) from the edge of element to element node.
\[ [A_w] = \frac{1}{3} \begin{bmatrix} (1+\phi_w) & 0 & 0 \\ 0 & (1+\phi_w) & 0 \\ 0 & 0 & (1+\phi_w) \end{bmatrix} \] (17)

\[ [A_s] = \frac{1}{3} \begin{bmatrix} \frac{2}{\lambda_s^2} & C_{s1} & S_{s1} \\ C_{s1} & \frac{2}{\lambda_s^2} & 0 \\ S_{s1} & 0 & 0 \end{bmatrix} \] (18)

Substituting (16) into (4), we define the bending curvatures as:
\[ \{\chi\} = [B_s]\{u_s\} \] (19)

\[ [B_s] = [B_{b_\eta}]+[B_{\phi b_\eta}][A_s]^{-1}[A_w] \]

Introducing (16) into (14) gives:
\[ \{\chi\} = [B_s]\{u_s\} \]
\[ [B_s] = [B_{b_\eta}][A_{\phi b}][A_w]; \quad [A_{\phi b}] = [A_w][A_s]^{-1} \] (20)

3 Formulation of MITC3

3.1 Bending strain matrix of MITC3

The displacement function is given as:
\[ w = \sum_{i=1}^{N_s} N_s w_i; \quad \beta_x = \sum_{i=1}^{N_s} N_s \beta_{x_i}; \quad \beta_y = \sum_{i=1}^{N_s} N_s \beta_{y_i} \] (21)

The relation between curvature and nodal displacement can be expressed in the equation below:
\[ \{\chi\} = [B_s]\{u_s\}; \quad [B_s] = [B_{b_\eta}] \] (22)

3.2 Shear strain interpolation

The key of MITC3 [11] technique is the use of appropriate assumed strain interpolation and the proper choice of tying points to relate the displacement interpolation and strain interpolation, which is done separately.

![Fig. 4. Tying points](image)

For MITC3, tying points [11-13] are chosen in the mid-points (Fig.4) of sides 1-2, 1-3, and 2-3. Distribution of \( \beta_x \) is assumed constant along \( \xi \) direction and \( \beta_\eta \) is assumed constant along \( \eta \):
\[ \beta_x = a_1 + a_2 \eta; \quad \beta_\eta = b_1 + b_2 \xi; \quad \beta_\lambda = \frac{1}{\sqrt{2}}(\beta_x - \beta_\eta) \] (23)

We obtain the values of \( \beta_x, \beta_\eta, \beta_\lambda \) at the tying points \( A, B, \) and \( C \):
- At point \( A (\eta=0) \), we obtain that \( a_1 = \beta_x(1,4) \);
- At point \( B (\xi=0) \), we obtain that \( b_1 = \beta_\eta(1,8) \);
- At node \( 2 (\xi=1, \eta=0) \), \( \beta_\lambda(1,0) = \frac{1}{\sqrt{2}}(\beta_x(1,0) - \beta_\eta(1,0)) \)
- At point \( 3 (\xi=0, \eta=1) \), \( \beta_\lambda(0,1) = \frac{1}{\sqrt{2}}(\beta_x(0,1) - \beta_\eta(0,1)) \)
- Along edge 3-2, \( \beta_x(1,0) = \beta_\lambda(0,1) \) leading to:
\[ \frac{1}{\sqrt{2}}(\beta_x(1,0) - \beta_\lambda(0,1)) = \frac{1}{\sqrt{2}}(\beta_\lambda(0,1) - \beta_\eta(0,1)) \]
Which gives: \( a_2 = -b_2 = c \)

Then the eq. (22) becomes:
\[ \beta_x = \beta_x(1,4) + c \eta; \quad \beta_\eta = \beta_\eta(1,8) - c \xi \] (24)

At point C (\( \xi=1/2, \eta=1/2 \)):
\[ \beta_x(C) = \frac{1}{\sqrt{2}}(\beta_x(C) - \beta_\eta(C)) \] and
\[ \beta_\eta(C) = \beta_\eta(1,8) - \beta_x(1,4) + \beta_\lambda(C) - \beta_\eta(C) \]

Substituting eq. (25) into (24), we obtain the expression:
\[ \beta_x = \beta_x(1,4) + (\beta_\eta + \beta_\lambda) - (\beta_\lambda + \beta_\eta) \]
\[ \beta_\eta = \beta_\eta(1,8) - \beta_x(1,4) + \beta_\lambda(C) - \beta_\eta(C) \]

\[ \beta_\lambda(C) = \frac{1}{\sqrt{2}}(\beta_x(C) - \beta_\eta(C)) \]

Substitute shear strain field in parametric space:
\[ \frac{\gamma_x}{\gamma_\eta} = \frac{w_x + b_1}{w_\eta + b_2}; \quad \frac{\gamma_\eta}{\gamma_x} = -\frac{w_\eta + b_2}{w_x + b_1} \] (28)

Then, from (26-28) we get the expression of shear strain in parametric space:
\[ \frac{\gamma_x}{\gamma_\eta} = \frac{w_x + b_1}{w_\eta + b_2}; \quad \frac{\gamma_\eta}{\gamma_x} = -\frac{w_\eta + b_2}{w_x + b_1} \] (29)

Transverse shear strain field in Cartesian space:
\[ \frac{\gamma_x}{\gamma_\eta} = \frac{w_x + b_1}{w_\eta + b_2}; \quad \frac{\gamma_\eta}{\gamma_x} = -\frac{w_\eta + b_2}{w_x + b_1} \] (30)

Where \( [J] \) is the inveres of Jacobian matrix.

Introducing (29) into (30), we obtain:
\[ \frac{\gamma_x}{\gamma_\eta} = \frac{w_x + b_1}{w_\eta + b_2}; \quad \frac{\gamma_\eta}{\gamma_x} = -\frac{w_\eta + b_2}{w_x + b_1} \] (31)

Mentioned by Jeon et al. in the article [14], MITC3 uses three-points Hammer integration to compute shear stiffness.

4 Numerical Analysis

To study the convergence behaviour of MITC3 and DKMT3 elements, a circular plate problem is analysed. Because of symmetry, only a quarter of circular plate as presents in (Fig.5) is evaluated. Two different boundary
conditions, i.e. a case of simply supported and a case of clamped supported in perimeter edge, are used with number of element (NELT) of 6, 24, 96, and 384.

![Circular plate with different numbers of elements](image)

Fig. 5. Circular plate with number of element NELT=24, $R = 5; h = 0.1, 1, 2, E = 10.92$; Poisson’s ratio $= 0.3$; and uniform load $f = 1$.

Table 1 presents the displacements at the centre of circle ($w_c$) for soft simply supported case, whereas Table 2 shows the results for clamped boundary condition. Fig. 6 and Fig. 7 show the convergence of $w_c$ of both elements.

<table>
<thead>
<tr>
<th>$R/h$</th>
<th>NELT</th>
<th>MITC3</th>
<th>DKMT</th>
<th>Exact [16]</th>
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<td>50</td>
<td>6</td>
<td>472.3</td>
<td>597.52</td>
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<td>24</td>
<td>3637</td>
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<tr>
<td>384</td>
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Table 2. $w$ for clamped supported boundary condition

<table>
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<td>384</td>
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</table>

5 Conclusion

We observe that both elements have good convergence towards the exact solution. DKMT element gives very good results and converges faster than MITC3 element. This is due to the quadratic interpolation functions for the rotations used in DKMT.

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References

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