An evaluation on the performance of two simple triangular bending plate elements

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Abstract. This paper will study and compare two different three-node triangular bending plate elements with three degree of freedom per node, i.e. MITC3 and DKMT. Both elements, which were developed based on Reissner-Mindlin plate theory and independent shear strain field, have simple formulation and have already been used widely. In this paper, numerical tests for circular plate case are conducted to verify the performance and show the convergence of these two triangular elements.

1 Introduction

In structural modelling, the use of triangular element is interesting due to its simplicity and flexibility. Three-node triangular element is mostly used for complex configuration. However, research on triangular is not as much as quadrilateral element. In finite element analysis, many researches show that quadrilateral element have a good performance. As a result, many analyses prefer to use the quadrilateral element [1-10]. This condition should encourage researchers to develop study about triangular element.

The challenge in finite element is how to generate a simple and applicable element formulation to reduce the computational cost, yet still has high accuracy and good convergence.

MITC (Mixed Interpolation of Tensorial Components) concept has been applied for triangular element. MITC3, proposed by Lee and Bathe [11], is one of the popular bending elements. There are quite a lot of studies and developments of MITC3 [12-15]. This three-node triangular element has a simple and general formulation. The behaviour of the element is isotropic, meaning that the stiffness matrix of the element does not depend on the sequence of node numbering.

Previously, Katili [16] proposed simple formulation for triangular bending plate element. The element, called DKMT, is an extension of DKT (Discrete Kirchhoff Triangular) element [17] which is a simple, efficient and reliable element for analysis of thin plates. In the case of thin plate problem, the solution obtained with DKMT element converged to that given by DKT element.

The purpose of this paper is to compare formulation and performance of two triangular elements, i.e. MITC3 and DKMT. Both of them have simple formulation. At the end of this paper, we conduct numerical studies for

circular bending plate to compare and evaluate the convergence of both elements.

2 Formulation of DKMT

The formulation of DKMT plate element is based on Reissner-Mindlin hypothesis, which only require C^0 continuity, and generated by assumed natural strain concept. Transversal shear deformation for this element is expressed with special interpolation called shear projection method. The triangular element has three-degrees of freedom per node (Fig.1).

Fig. 1. Triangle element and the degrees of freedom

2.1 Bending strain matrix

The DKMT element was first published in 1993 [16]. DKMT element has 3 nodes with 3 degrees of freedom each, which are: w (translation in the z direction), β_x (rotation in the z - x plane) and β_y (rotation in the z - y plane). Incomplete quadratic rotation fields for β_x and β_y are considered in terms of rotations at the three corners and a temporary variable at mid-side i - j (Fig.2).

The displacement function is given as:

$$w = \sum_{i=1}^3 N_i w_i$$

$$\beta_x = \sum_{i=1}^3 N_i \beta_{x_i} + \sum_{k=4}^6 P_k C_k \Delta \beta_{s_k} \quad (1)$$

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$$\beta_y = \sum_{i=1}^3 N_i \beta_{y_i} + \sum_{k=4}^6 P_k C_k \Delta \beta_{s_k}$$

where: w_i is the vertical displacement at node- i
 N_i is the shape function at node- i
 β_{x_i}, β_{y_i} are the rotation in plane of z - x and z - y

Fig. 2. Kinematic variable of corner node and temporary variable at mid side i - j .

N_i are the linear shape functions and P_k is the quadratic functions:

$$N_1 = \lambda \quad ; \quad N_2 = \xi \quad ; \quad N_3 = \eta \quad ; \quad \lambda = 1 - \xi - \eta \quad (2)$$

$$P_4 = 4\lambda\xi \quad ; \quad P_5 = 4\xi\eta \quad ; \quad P_6 = 4\lambda\eta$$

C_k and S_k are the direction cosines of side i - j (Fig.2)

$$\{\chi\} = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{Bmatrix} \quad (3)$$

$\beta_{x,x}$ and $\beta_{y,y}$ denote the first derivatives of β_x with respect to x and y , respectively.

The relation between curvature and nodal displacement is expressed in the equation below.

$$\{\chi\} = [B_{bp}] \{u_n\} + [B_{b\Delta\beta}] \{\Delta\beta_{s_n}\} \quad (4)$$

where: $\{u_n\} = \langle u_n \rangle^T = \langle \dots w_i \quad \beta_{x_i} \quad \beta_{y_i} \dots \rangle_{i=1,2,3}^T$

$$\{\Delta\beta_{s_n}\} = \langle \Delta\beta_{s_n} \rangle^T = \langle \Delta\beta_{s_4} \quad \Delta\beta_{s_5} \quad \Delta\beta_{s_6} \rangle^T$$

From eq. (1-4), we obtain the expression of bending strain matrix,

$$[B_{bp}] = \begin{bmatrix} 0 & N_{i,x} & 0 \\ \dots & 0 & 0 & N_{i,y} & \dots \\ 0 & N_{i,y} & N_{i,x} \end{bmatrix}_{i=1,2,3} \quad (5)$$

$$[B_{b\Delta\beta}] = \begin{bmatrix} P_{k,x} C_k \\ \dots & P_{k,y} S_k & \dots \\ P_{k,y} C_k + P_{k,x} S_k \end{bmatrix}_{k=4,5,6} \quad (6)$$

$N_{i,x}$, $P_{k,y}$, $N_{i,x}$ and $P_{k,y}$ denote the first derivatives of N_i and P_k with respect to x and y , respectively.

2.2 Shear strain interpolation

Shear deformation field is approximated linear in each element.

$$\{\gamma\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = \sum_{i=1}^3 N_i \begin{Bmatrix} \gamma_{x_i} \\ \gamma_{y_i} \end{Bmatrix} \quad (7)$$

where: γ_{x_i} and γ_{y_i} are the shear deformations at nodes- i

Shear deformation is assumed constant along the edge of element (Fig.3). Shear deformation at node- i is obtained from projection of constant shear deformation γ_{s_k} from the edge of element to element node.

$$\begin{Bmatrix} \gamma_{s_4} \\ \gamma_{s_6} \end{Bmatrix} = \begin{bmatrix} C_{12} & S_{12} \\ C_{31} & S_{31} \end{bmatrix} \begin{Bmatrix} \gamma_{x_1} \\ \gamma_{y_1} \end{Bmatrix} ; \quad \begin{Bmatrix} \gamma_{s_5} \\ \gamma_{s_4} \end{Bmatrix} = \begin{bmatrix} C_{23} & S_{23} \\ C_{12} & S_{12} \end{bmatrix} \begin{Bmatrix} \gamma_{x_2} \\ \gamma_{y_2} \end{Bmatrix} \quad (8)$$

$$\begin{Bmatrix} \gamma_{s_6} \\ \gamma_{s_5} \end{Bmatrix} = \begin{bmatrix} C_{31} & S_{31} \\ C_{23} & S_{23} \end{bmatrix} \begin{Bmatrix} \gamma_{x_3} \\ \gamma_{y_3} \end{Bmatrix}$$

Fig. 3. Constant transverse shear strain along edge ij for DKMT

where:

$$C_{ij} = \frac{x_{ji}}{L_{ij}} \quad ; \quad S_{ij} = \frac{y_{ji}}{L_{ij}} \quad ; \quad L_{ij} = \sqrt{x_{ji}^2 + y_{ji}^2} \quad (9)$$

$$x_{ji} = x_j - x_i \quad ; \quad y_{ji} = y_j - y_i$$

From eq. (7-9), we obtain:

$$\{\gamma\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = [B_{s_\gamma}] \{\gamma_{s_n}\} \quad (10)$$

with: $\{\gamma_{s_n}\} = \langle \gamma_{s_n} \rangle^T = \langle \gamma_{s_4} \quad \gamma_{s_5} \quad \gamma_{s_6} \rangle^T$

$$[B_{s_\gamma}] = \begin{bmatrix} \left(\frac{S_{31}\lambda - S_{23}\xi}{A_1} \right) & \left(\frac{S_{12}\xi - S_{31}\eta}{A_2} \right) & \left(\frac{S_{23}\xi - S_{12}\lambda}{A_3} \right) \\ \left(\frac{C_{23}\xi - C_{31}\lambda}{A_2} \right) & \left(\frac{C_{31}\eta - C_{12}\xi}{A_3} \right) & \left(\frac{C_{12}\lambda - C_{23}\eta}{A_1} \right) \end{bmatrix} \quad (11)$$

where:

$$A_1 = C_{12} S_{31} - C_{31} S_{12} \quad ; \quad A_2 = C_{23} S_{12} - C_{12} S_{23} \quad ; \quad A_3 = C_{31} S_{23} - C_{23} S_{31}$$

Katili [17] proposed the assumed independent transverse shear strain along the side i - j expressed as:

$$\gamma_{s_k} = -\frac{2}{3} \phi_k \Delta \beta_{s_k} \quad ; \quad \phi_k = \frac{2}{\kappa(1-\nu)} \left(\frac{h^2}{L_{ij}^2} \right) \quad (12)$$

Where h is the thickness of element and κ is the shear correction factor (usually $\kappa = 5/6$).

If (12) is applied on all sides of the element, the following matrix relation is obtained:

$$\{\gamma_{s_n}\} = [A_\phi] \{\Delta\beta_{s_n}\} \quad ; \quad [A_\phi] = -\frac{2}{3} \begin{bmatrix} \phi_4 & 0 & 0 \\ 0 & \phi_5 & 0 \\ 0 & 0 & \phi_6 \end{bmatrix} \quad (13)$$

Substituting (13) into (12) we get:

$$\{\gamma\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = [B_{s_\gamma}] [A_\phi] \{\Delta\beta_{s_n}\} \quad (14)$$

If the assumed shear force and strains are constant on the side, then we obtain:

$$\gamma_{s_k} = \frac{1}{L_{ij}} \int_0^{L_{ij}} \gamma_s ds = -\frac{2}{3} \phi_k \Delta \beta_{s_k} \quad ; \quad \gamma_s = w,s + \beta_s \quad (15)$$

Applying (15) on each side- ij , we obtain

$$\{\Delta\beta_{s_n}\} = [A_\Delta]^{-1} [A_u] \{u_n\} \quad \text{where} \quad (16)$$

$$[A_\Delta] = -\frac{2}{3} \begin{bmatrix} (1+\phi_4) & 0 & 0 \\ 0 & (1+\phi_5) & 0 \\ 0 & 0 & (1+\phi_6) \end{bmatrix} \quad (17)$$

$$[A_u] = \frac{1}{2} \begin{bmatrix} -\frac{2}{L_{12}} & C_{12} & S_{12} & \frac{2}{L_{12}} & C_{12} & S_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{L_{23}} & C_{23} & S_{23} & \frac{2}{L_{23}} & C_{23} & S_{23} \\ \frac{2}{L_{31}} & C_{31} & S_{31} & 0 & 0 & 0 & -\frac{2}{L_{31}} & C_{31} & S_{31} \end{bmatrix} \quad (18)$$

Substituting (16) into (4), we define the bending curvatures as:

$$\{\chi\} = [B_b] \{u_n\} \quad (19)$$

$$[B_b] = [B_{b_p}] + [B_{b_{\Delta p}}] [A_\Delta]^{-1} [A_u]$$

Introducing (16) into (14) gives:

$$\{\gamma\} = [B_s] \{u_n\} \quad (20)$$

$$[B_s] = [B_{s_\gamma}] [A_{\phi\Delta}] [A_u] ; [A_{\phi\Delta}] = [A_\phi] [A_\Delta]^{-1}$$

3 Formulation of MITC3

3.1 Bending strain matrix of MITC3

The displacement function is given as:

$$w = \sum_{i=1}^3 N_i w_i ; \beta_x = \sum_{i=1}^3 N_i \beta_{x_i} ; \beta_y = \sum_{i=1}^3 N_i \beta_{y_i} \quad (21)$$

The relation between curvature and nodal displacement can be expressed in the equation below.

$$\{\chi\} = [B_b] \{u_n\} ; [B_b] = [B_{b_p}] \quad (22)$$

3.2 Shear strain interpolation

The key of MITC3 [11] technique is the use of appropriate assumed strain interpolation and the proper choice of tying points to relate the displacement interpolation and strain interpolation, which is done separately.

Fig. 4. Tying points

For MITC3, tying points [11-13] are chosen in the mid-points (Fig.4) of sides 1-2, 1-3, and 2-3. Distribution of β_ξ is assumed constant along ξ direction and β_η is assumed constant along η :

$$\beta_\xi = a_1 + a_2 \eta ; \beta_\eta = b_1 + b_2 \xi ; \beta_\lambda = \frac{1}{\sqrt{2}} (\beta_\xi - \beta_\eta) \quad (23)$$

We obtain the values of β_ξ , β_η , β_λ at the tying points A, B, and C.

– At point A ($\eta=0$), we obtain that $a_1 = \beta_{\xi(A)}$.

– At point B ($\xi=0$), we obtain that $b_1 = \beta_{\eta(B)}$.

– At node 2 ($\xi=1, \eta=0$), $\beta_\lambda(1,0) = \frac{1}{\sqrt{2}} (\beta_\xi(1,0) - \beta_\eta(1,0))$

– At node 3 ($\xi=0, \eta=1$), $\beta_\lambda(0,1) = \frac{1}{\sqrt{2}} (\beta_\xi(0,1) - \beta_\eta(0,1))$

– Along edge 3-2, $\beta_\lambda(1,0) = \beta_\lambda(0,1)$ leading to:

$$\frac{1}{\sqrt{2}} (\beta_\xi(1,0) - \beta_\eta(1,0)) = \frac{1}{\sqrt{2}} (\beta_\xi(0,1) - \beta_\eta(0,1))$$

Which gives: $a_2 = -b_2 = c$

Then the eq. (22) becomes:

$$\beta_\xi = \beta_{\xi(A)} + c \eta ; \beta_\eta = \beta_{\eta(B)} - c \xi \quad (24)$$

– At point C ($\xi=1/2, \eta=1/2$):

$$\beta_{\lambda(C)} = \frac{1}{\sqrt{2}} \{ \beta_{\xi(C)} - \beta_{\eta(C)} \} \quad \text{and}$$

$$\beta_{\lambda(C)} = \frac{1}{\sqrt{2}} \left\{ \left(\beta_{\xi(A)} + \frac{1}{2} c \right) - \left(\beta_{\eta(B)} - \frac{1}{2} c \right) \right\}$$

and we obtain the value of c,

$$c = \beta_{\eta(B)} - \beta_{\xi(A)} + \beta_{\xi(C)} - \beta_{\eta(C)} \quad (25)$$

Substituting eq. (25) into (24), we obtain the expression,

$$\beta_\xi = \beta_{\xi(A)} + \left(\beta_{\eta(B)} - \beta_{\xi(A)} + \beta_{\xi(C)} - \beta_{\eta(C)} \right) \eta \quad (26)$$

$$\beta_\eta = \beta_{\eta(B)} - \left(\beta_{\eta(B)} - \beta_{\xi(A)} + \beta_{\xi(C)} - \beta_{\eta(C)} \right) \xi$$

β_ξ and β_η at tying points are average value of two points at node.

$$\beta_{\xi(A)} = \frac{1}{2} (\beta_{\xi_1} + \beta_{\xi_2}) ; \beta_{\eta(B)} = \frac{1}{2} (\beta_{\eta_1} + \beta_{\eta_3}) \quad (27)$$

$$\beta_{\xi(C)} = \frac{1}{2} (\beta_{\xi_2} + \beta_{\xi_3}) ; \beta_{\eta(C)} = \frac{1}{2} (\beta_{\eta_2} + \beta_{\eta_3})$$

Transverse shear strain field in parametric space:

$$\begin{Bmatrix} \gamma_\xi \\ \gamma_\eta \end{Bmatrix} = \begin{Bmatrix} w_{,\xi} + \beta_\xi \\ w_{,\eta} + \beta_\eta \end{Bmatrix} = \begin{Bmatrix} w_2 - w_1 + \beta_\xi \\ w_3 - w_1 + \beta_\eta \end{Bmatrix} \quad (28)$$

Then, from (26-28) we get the expression of shear strain in parametric space:

$$\begin{Bmatrix} \gamma_\xi \\ \gamma_\eta \end{Bmatrix} = [B_{s_\xi}] \{u_n\} \quad (29)$$

$$[B_{s_\xi}] = \frac{1}{2} \begin{bmatrix} -2 & (1-\eta) & \eta & 2 & 1 & -\eta & 0 & \eta & 0 \\ -2 & \xi & (1-\xi) & 0 & 0 & \xi & 2 & -\xi & 1 \end{bmatrix}$$

Transverse shear strain field in Cartesian system:

$$\{\underline{\gamma}\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = [j] \begin{Bmatrix} \gamma_\xi \\ \gamma_\eta \end{Bmatrix} \quad (30)$$

Where [j] is the invers of Jacobian matrix.

Introducing (29) into (30), we obtain:

$$\{\underline{\gamma}\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = [B_s] \{u_n\} ; [B_s] = [j] [B_{s_\xi}] \quad (31)$$

Mentioned by Jeon *et al.* in the article [14], MITC3 uses three-points Hammer integration to compute shear stiffness.

4 Numerical Analysis

To study the convergence behaviour of MITC3 and DKMT elements, a circular plate problem is analysed. Because of symmetry, only a quarter of circular plate as presents in (Fig.5) is evaluated. Two different boundary

conditions, i.e. a case of simply supported and a case of clamped supported in perimeter edge, are used with number of element (NELT) of 6, 24, 96, and 384.

Fig. 5. Circular plate with number of element NELT=24. $R = 5$; $h = 0.1, 1, 2$; $E = 10.92$; Poisson's ratio $\nu = 0.3$; and uniform load $f_z = 1$.

Table 1 presents the displacements at the centre of circle (w_c) for soft simply supported case, whereas Table 2 shows the results for clamped boundary condition. Fig. 6 and Fig. 7 show the convergence of w_c of both elements.

Table 1. w_c for simply supported boundary condition

R/h	NELT	MITC3	DKMT	Exact [16]
50	6	29056	37780	39831
	24	36374	39353	
	96	38986	39716	
	384	39690	39802	
5	6	35.019	39.404	41.599
	24	39.705	41.038	
	96	41.143	41.459	
	384	41.491	41.567	
2.5	6	5.044	5.554	5.870
	24	5.636	5.793	
	96	5.814	5.852	
	384	5.857	5.866	

Table 2. w_c for clamped supported boundary condition

R/h	NELT	MITC3	DKMT	Exact [16]
50	6	472.3	9975.2	9783.5
	24	4381.4	9991.4	
	96	8944.1	9841.2	
	384	9681.9	9796.0	
5	6	6.976	11.601	11.551
	24	10.341	11.694	
	96	11.271	11.587	
	384	11.484	11.560	
2.5	6	1.564	2.079	2.114
	24	1.969	2.127	
	96	2.080	2.118	
	384	2.106	2.115

Fig. 6. Convergence of w_c for simply supported case

Fig. 7. Convergence of w_c for clamped case

5 Conclusion

We observe that both elements have good convergence towards the exact solution. DKMT element gives very good results and converges faster than MITC3 element. This is due to the quadratic interpolation functions for the rotations used in DKMT.

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