A study on dynamic response of functionally graded sandwich beams under different dynamic loadings

Wachirawit Songsuwan^{1*}, Monsak Pimsarn¹, and Nuttawit Wattanasakulpong²,

¹Mechanical Engineering Department, Faculty of Engineering, King Mongkut's Institute of Technology Ladkrabang, Bangkok, Thailand

²Mechanical Engineering Department, Faculty of Engineering, Mahanakorn University of Technology, Nongchok, Bangkok, Thailand

Abstract. In this research, free and forced vibration of functionally graded sandwich beams is considered using Timoshenko beam theory which takes into account the significant effects of transverse shear deformation and rotary inertia. The governing equations of motion are formulated from Lagrange's equations and they are solved by using The Ritz and Newmark methods. The results are presented in both tabular and graphical forms to show the effects of layer thickness ratios, boundary conditions, length to height ratios, etc. on natural frequencies and dynamic deflections of the beams. According to the numerical results, all parametric studies considered in this research have significant impact on free and forced behaviour of the beams; for example, the frequency is low and the dynamic deflection is large for the beams which are hinged at both ends.

1 Introduction

Recently, functionally graded (FG) sandwich structures play a vital role in modern structural engineering with excellent properties in high strength-to-weight ratio. The FG sandwich structures have no problems of de-bonding and delaminating modes of failure between layers which often occur in conventional sandwich structures due to the mismatch of materials at the interface. In the literature, buckling and vibration of FG sandwich beams with the combination of general and non-general boundary conditions were considered by Tossapanon and Wattanasakulpong [1] and Trinh et al. [2]. Moreover, in the study of Nguyen et al. [3], they also provided the solutions of natural frequencies and critical buckling load of such beams. A quasi-3D theory was employed to deal with static bending, buckling and vibration problems of FG sandwich beams in Refs. [4-5]. Based on the literature, most of previous studies presented only the static and free vibration analysis of FG sandwich beams. In the study of Vo et al. [6], the development of finite element model was presented for vibration and buckling of FG sandwich beams under various general boundary conditions. The relationship between fundamental natural frequency and in-plane loading was also presented in the study.

This investigation aims to extend research work to deal with dynamic behaviour of FG sandwich beams under various kinds of different dynamic loadings. Ritz method incorporated with Newmark time integration procedure are used to solve equations of motion based on Timoshenko beam theory. Many important parametric

studies such as boundary condition, layer thickness ratio, material volume fraction index, etc. are investigated.

2 FG Sandwich beams

A geometry of FG sandwich beam including three layers of FG face sheets and metal homogenous core is shown in Fig. 1. The FG face sheets are the composite materials with the graded mix of ceramic and metal phases. The three numeric notations are used to represent the layer thickness ratio. For example, 1-1-1 indicates that the beam is composed of equal thickness in each layer.

Fig. 1 Geometry and coordinate of FG sandwich beam.

The equations for estimating the effective material properties of the beam can be expressed as follows [1]:

$$E^{(i)}(z) = (E_b - E_t)V_b^{(i)}(z) + E_t, \quad (1)$$

$$\rho^{(i)}(z) = (\rho_b - \rho_t)V_b^{(i)}(z) + \rho_t, \quad (2)$$

where $E^{(i)}(z)$ and $\rho^{(i)}(z)$ are the Young's modulus and material density in each layer. The positions of material

* Corresponding author: w.songsuwan@gmail.com

properties at the faces and the core are denoted by the subscripts t and b respectively. The constant value of Poisson's ratio (ν) is used in this investigation. The material volume fraction, ($V_b^{(i)}$), related to the power law distribution can be expressed as follows [1]:

$$\begin{cases} V_b^{(1)}(z) = \left(\frac{z - h_0}{h_1 - h_0} \right)^n & z \in [h_0, h_1] \\ V_b^{(2)}(z) = 1 & z \in [h_1, h_2] \\ V_b^{(3)}(z) = \left(\frac{z - h_3}{h_2 - h_3} \right)^n & z \in [h_2, h_3] \end{cases} \quad (3)$$

where n is the material volume fraction index or power law index, $0 \leq n \leq \infty$. For the beam in Fig. 1, we use ($E_b = E_m$, $\rho_b = \rho_m$) and ($E_t = E_c$, $\rho_t = \rho_c$). It is also noted that the subscripts c and m are used to define the material properties of ceramic and metal phases, respectively.

3 Mathematical Formulations

The dynamic models of FG sandwich beams are formulated in accordance with Timoshenko beam theory [1]. The displacements of any point of the beams along the x - and z -axes, which are denoted by $u(x, z, t)$, $w(x, z, t)$, respectively, can be written below:

$$u(x, z, t) = u_0(x, t) + z\psi(x, t), \quad (4)$$

$$w(x, z, t) = w_0(x, t), \quad (5)$$

where u_0 and w_0 are axial and transverse displacements in the middle plane ($z=0$), respectively, ψ is the rotation of the beam cross-section and t is time. The strain-displacement relations in terms of normal strain (ε_{xx}) and shear strain (γ_{xz}) are given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z \frac{\partial \psi}{\partial x}, \quad (6)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial w_0}{\partial x} + \psi. \quad (7)$$

The corresponding normal stress (σ_{xx}) and shear stress (τ_{xz}) can be obtained from the elastic constitutive law as

$$\sigma_{xx} = E(z)\varepsilon_{xx}, \quad (8)$$

$$\tau_{xz} = G(z)\gamma_{xz} = \frac{E(z)}{2(1+\nu)}\gamma_{xz}. \quad (9)$$

The strain energy (U) of the FG sandwich beams at any instant can be defined in form of normal stress (σ_{xx}), normal strain (ε_{xx}), shear stress (τ_{xz}) and shear strain (γ_{xz}) as

$$U = \frac{b}{2} \int_0^L \int_{-h/2}^{h/2} (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz}) dz dx. \quad (10)$$

For the kinetic energy (K), it is expressed in form of displacements of u and w as

$$K = \frac{b}{2} \int_0^L \int_{-h/2}^{h/2} \rho(z) \left[(\dot{u})^2 + (\dot{w})^2 \right] dz dx. \quad (11)$$

The work done (V) by the dynamic load (P) is $V = P(t)w(L/2)$ and the total energy of the system is

$$\Pi = K + V - U. \quad (12)$$

The total energy can be solved by using the Ritz and Newmark method [7]. The implementation of such methods to free and forced vibration of the beams was described elsewhere in details [8]. An example of the trial displacement functions in form of algebraic polynomials for a case of beam clamped at both ends is:

$$\begin{cases} u_0(x, t) = \sum_{j=1}^N A_j(t) \left(\frac{x}{L} \right)^j \left(1 - \frac{x}{L} \right) \\ w_0(x, t) = \sum_{j=1}^N B_j(t) \left(\frac{x}{L} \right)^j \left(1 - \frac{x}{L} \right) \\ \psi(x, t) = \sum_{j=1}^N C_j(t) \left(\frac{x}{L} \right)^j \left(1 - \frac{x}{L} \right) \end{cases} \quad (13)$$

4 Results

In this research, FG sandwich beams having thickness ($h=1$ m), breadth ($b=0.5$ m) and length ($L=20$ m) are made of a mixture of Alumina (Al_2O_3) as ceramic phases and Aluminum (Al) as metal phases whose material properties such as Young's modulus (E), Poisson's ratio (ν) and material density (ρ) are:

For Alumina (Al_2O_3)

$$E_c = 380 \text{ GPa}, \nu_c = 0.3, \rho_c = 3960 \text{ kg/m}^3$$

For Aluminum (Al)

$$E_m = 70 \text{ GPa}, \nu_m = 0.3, \rho_m = 2702 \text{ kg/m}^3$$

Table 1 shows the convergence study and validation of dimensionless frequencies of the beams with hinged at both ends (H-H). Accuracy is confirmed by the result of Ref. [6] for the case of the beam with material volume fraction index ($n=0.5$). N is the polynomial number of terms in the Ritz method.

Table 1 Convergence study and validation for free vibration of 1-1-1 FG sandwich beams.

N	ω_1	ω_2	ω_3
4	4.6405	22.9665	57.4058
6	4.6391	18.3272	40.6471
8	4.6391	18.2530	40.0224
9	4.6391	18.2527	40.0223
10	4.6391	18.2527	40.0223
[6]	4.6294	-	-

Fig. 2 Dimensionless fundamental frequency of FG sandwich beams.

To study an influence of the beam thickness ratio (L/h) on fundamental frequency of FG sandwich beams ($n=0.5$, H-H), Fig. 2 shows the plot of the 1st frequency of the beams with various layer thickness ratios. It can be observed that the frequency increases significantly as the increase of L/h ratio and the frequency of 2-1-2 is higher than that of other beams considered in this research.

In the following investigations, we consider the forced vibration of FG sandwich beams under Heaviside step loading and harmonic loading as shown in Fig. 3.

Fig. 3 Heaviside step and harmonic loadings.

All dynamic deflections are measured at the middle position of the beams. Fig. 4 and Fig. 5 show the effects of the index n and boundary condition on dynamic deflections of 1-1-1 FG sandwich beams under the Heaviside step loading, respectively. The dynamic deflection in this research is presented in the non-dimensional form of $w(x,t)/w_s$ where $w(x,t)$ is dynamic deflection and w_s is static deflection, $w_s = PL^3/48E_m I$ where $I = bh^3/12$. As can be seen, the beam with low value of n , $n < 1.0$, (percentage of metal more than that of ceramic) is soft with larger dynamic deflection throughout the time. $n=1.0$ means that the percentage of metal is equal to the ceramic; while $n > 1.0$, the beam is composed of much more ceramic than metal. For the effect of boundary condition, it is observed that the beam hinged at both ends (H-H) has larger deflection than that of the beam which is clamped at one end and hinged at another one defined as C-H beam. And the beam

clamped at both ends (C-C) shows its lowest deflection through the considered time. Moreover, when the beams subjected to the Heaviside step loading, we can obtain the dynamic deflections for both areas which are the area of forced vibration ($t = 0 \rightarrow 0.5$ s) and free vibration ($t = 0.5 \rightarrow 1.0$ s).

Fig. 4 Dynamic deflections of FG sandwich beams supported by H-H under Heaviside step loading: Effect of n .

Fig. 5 Dynamic deflections of FG sandwich beams with $n=0.5$ under Heaviside step loading: Effect of boundary condition.

In Fig.6, the beams with various layer thickness ratios are investigated to find out their dynamic deflections under the action of harmonic loading. The boundary condition of the beams is H-H and the material volume fraction is $n=0.5$. From this figure, the deflection of the beam with 2-5-3 ratio is larger than that of other beams

To consider the significant effect of boundary condition on dynamic response of FG sandwich beams under the harmonic loading, Fig. 7 shows the dynamic deflections of beams with different supports. The beams having 1-1-1 and $n=0.5$ are considered. It is clearly seen that the beam, which is clamped at both ends (C-C), is very strong and gives less deflection throughout the considered time as compared to the deflections of C-H and H-H beams, respectively.

Fig. 6 Dynamic deflections of FG sandwich beams under harmonic loading: Effect of layer thickness ratio.

Fig. 7 Dynamic deflections of FG sandwich beams under harmonic loading: Effect of boundary condition.

5 Conclusions

The free and forced vibration responses of FG sandwich beams under Heaviside step and harmonic loadings are investigated using the Ritz and Newmark methods. Within the framework of Timoshenko beam theory including the effects of shear deformation and rotary inertia, the equation of motion is established for the beams with different boundary conditions. The influences of parametric studies such as layer thickness ratio, boundary condition, length to height ratio, material volume fraction index of the beams are presented and discussed in detail. The present modelling is validated and accurate with the comparison between the obtained solutions and the previous ones in the literature. Numerical results reveal that the beams which are clamped at both ends (C-C) have higher natural frequencies and lower dynamic deflections than those of C-H and H-H beams, respectively. Additionally, the dynamic deflection decreases as the increase of parameter n . It is due to material properties of the beams are dependent on the parameter.

References

1. P. Tossapanon, N. Wattanasakulpong, *Compos Struc.* **142**, 215-225 (2016)
2. L.C. Trinh, T.P. Vo, A.I. Osofero, J. Lee, *Compos Struc.* **156**, 263-275 (2016)
3. T.K. Nguyen, T.T.P. Nguyen, T.P. Vo, H.T. Thai, *Compos Part B.* **76**, 237-285 (2015)
4. T.P. Vo, H.T. Thai, T.K. Nguyen, F. Inam, J. Lee, *Compos Struc.* **119**, 1-12 (2015)
5. T.P. Vo, H.T. Thai, T.K. Nguyen, F. Inam, J. Lee, *Compos Part B* **68**, 59-74 (2015)
6. T.P. Vo, H.T. Thai, T.K. Nguyen, A. Maheri, J. Lee, *Eng Struc* **64**, 12-22 (2014)
7. D. Chen, J. Yang, S. Kitipornchai, *Int J Mech Sci* **108-109**, 14-22 (2016)
8. W. Songsuwan, M. Pimsarn, N. Wattanasakulpong, *In. J. Struct. Stab. Dyn.* **18**(9), 1850112 (2018)