Γ -Paired dominating graphs of some paths

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Abstract. A paired dominating set of a graph $G = (V(G), E(G))$ is a set D of vertices of G such that every vertex is adjacent to some vertex in D , and the subgraph of G induced by D contains a perfect matching. The upper paired domination number of G , denoted by $\Gamma_{pr}(G)$, is the maximum cardinality of a minimal paired dominating set of G . A paired dominating set of cardinality $\Gamma_{pr}(G)$ is called a $\Gamma_{pr}(G)$ -set. The Γ -paired dominating graph of G , denoted by $PD_{\Gamma}(G)$, is the graph whose vertex set is the set of all $\Gamma_{pr}(G)$ -sets, and two $\Gamma_{pr}(G)$ -sets are adjacent in $PD_{\Gamma}(G)$ if one can be obtained from the other by removing one vertex and adding another vertex of G . In this paper, we present the Γ -paired dominating graphs of some paths.

1 Introduction

We use [1, 2] for the terminology and notation which are not defined here. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set of a graph G is a set $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex in D . The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . A dominating set D is a $\gamma(G)$ -set in G if $|D| = \gamma(G)$. For a detailed literature on domination in graphs, the reader can refer to [3] and [4].

In 2010, Lakshmanan and Vijayakumar [5] defined the gamma graph $\gamma.G$ of G , which is the graph whose vertex set is the set of all $\gamma(G)$ -sets, and $\gamma(G)$ -sets D_1 and D_2 are adjacent in $\gamma.G$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = D_2 \setminus \{u\} \cup \{v\}$. They studied the gamma graphs of cographs and graph products.

In 2011, Fricke et al. [6] also defined the gamma graph $G(\gamma)$ of G , which is the graph with the same vertex set as $\gamma.G$, and two $\gamma(G)$ -sets D_1 and D_2 are adjacent in $G(\gamma)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = D_2 \setminus \{u\} \cup \{v\}$, and u and v have to be adjacent in G . Notice that $G(\gamma)$ is a spanning subgraph of $\gamma.G$.

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In 2013, Haas and Seyffarth [7] defined the k -dominating graph of G , denoted by $D_k(G)$. Its vertex set contains all dominating sets of size at most k . Two vertices are adjacent in $D_k(G)$ if the corresponding dominating sets of G differ by either adding or deleting a single vertex of G . The authors gave conditions that ensure $D_k(G)$ is connected.

In 2017, Wongsriya and Trakultraipruk [8] introduced the γ -total dominating graph of G , denoted by $TD_\gamma(G)$, is the graph whose vertices are γ -total dominating sets, and two γ -total dominating sets D_1 and D_2 are adjacent in $TD_\gamma(G)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = D_2 \setminus \{u\} \cup \{v\}$. The authors presented the γ -total dominating graphs of paths and cycles.

A *matching* in a graph G is a set of independent edges in G . A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M . Haynes and Slater [9] introduced the concept of paired domination. A *paired dominating set* of G is a dominating set whose induced subgraph has a perfect matching. A paired dominating set D of G is *minimal* if no proper subset of D is a paired dominating set of G . The maximum cardinality of a minimal paired dominating set of G is called the *upper paired domination number* of G , denoted by $\Gamma_{pr}(G)$. We use the term $\Gamma_{pr}(G)$ -set to refer to a paired dominating set of cardinality $\Gamma_{pr}(G)$.

In this paper, we define the Γ -paired dominating graph of a graph G , denoted by $PD_\Gamma(G)$, be the graph whose vertex set is the set of all $\Gamma_{pr}(G)$ -sets, and two $\Gamma_{pr}(G)$ -sets D_1 and D_2 are adjacent in $PD_\Gamma(G)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = D_2 \setminus \{u\} \cup \{v\}$. For instance, the Γ -paired dominating graphs of the paths $P_4 = v_1v_2v_3v_4$ and $P_7 = v_1v_2v_3v_4v_5v_6v_7$ are shown in figures 1 and 2, respectively.

Fig. 1. The Γ -paired dominating graph of P_4 .

Fig. 2. The Γ -paired dominating graph of P_7 .

2 Γ -Paired dominating graphs of some paths

In this section, we present the Γ -paired dominating graphs of some paths. We always let $P_n = v_1v_2 \cdots v_n$ be the path with n vertices. If $n = 1$, then there are no $\Gamma_{pr}(P_1)$ -sets, so $PD_\Gamma(P_1)$ is the null graph, i.e., its vertex set and edge set are empty. If $n = 2$, then there is only one $\Gamma_{pr}(P_2)$ -set, which is $\{v_1, v_2\}$, so $PD_\Gamma(P_2) \cong K_1$. We then consider the Γ -paired dominating graphs of P_n where $3 \leq n \leq 13$.

Dorbec et al. [10] established the upper paired domination numbers of all paths as shown in the following lemma.

Lemma 2.1. *If $n \geq 2$ is an integer, then $\Gamma_{pr}(P_n) = 8 \left\lfloor \frac{n+1}{11} \right\rfloor + 2 \left\lfloor \frac{(n+1) \bmod 11}{3} \right\rfloor$.*

We first consider the Γ -paired dominating graphs of P_3 and P_5 . Then we get the following theorem.

Theorem 2.2. *If $k \in \{1, 2\}$, then $PD_{\Gamma}(P_{2k+1}) \cong K_{k+1}$.*

Proof. By Lemma 2.1, we have $\Gamma_{pr}(P_3)=2$ and $\Gamma_{pr}(P_5)=4$. If $k = 1$, then there are two $\Gamma_{pr}(P_3)$ -sets, which are $\{v_1, v_2\}$ and $\{v_2, v_3\}$, so $PD_{\Gamma}(P_3) \cong K_2$. If $k = 2$, then there are three $\Gamma_{pr}(P_5)$ -sets, which are $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_4, v_5\}$, and $\{v_2, v_3, v_4, v_5\}$, so $PD_{\Gamma}(P_5) \cong K_3$. $\square$

Let G and H be graphs. The *Cartesian product* of G and H , denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ whose vertices (u, v) and (u', v') are adjacent if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

Let K_p be the complete graph with vertices $u_1, u_2, \dots, u_p$ and K_q the complete graph with vertices $v_1, v_2, \dots, v_q$. We define the *steprook* $SR(p, q)$ be the subgraph of $K_p \square K_q$ induced by $\{(u_x, v_y) : x - y \leq 1\}$.

For instance, the steprooks $SR(1,1)$ and $SR(3,3)$ are shown in figures 3 and 4, respectively.

Fig. 3. The steprook $SR(1,1)$.

Fig. 4. The steprook $SR(3,3)$.

Theorem 2.3. *If $k \in \{2, 3, 4, 5\}$, then $PD_{\Gamma}(P_{2k}) \cong SR(k-1, k-1)$.*

Proof. If $k = 2$, then there is only one $\Gamma_{pr}(P_4)$ -set, which is $\{v_2, v_3\}$, so $PD_{\Gamma}(P_4) \cong SR(1,1)$. We consider the Γ -paired dominating graph of P_6 . By Lemma 2.1, we have $\Gamma_{pr}(P_6)=4$. Then the following are all $\Gamma_{pr}(P_6)$ -sets: $D_1 = \{v_1, v_2, v_4, v_5\}$, $D_2 = \{v_1, v_2, v_5, v_6\}$, $D_3 = \{v_2, v_3, v_4, v_5\}$, and $D_4 = \{v_2, v_3, v_5, v_6\}$. Then we have $PD_{\Gamma}(P_6) \cong SR(2,2)$ as shown in figure 5.

Fig. 5. The steprook $SR(2,2)$.

We next consider the Γ -paired dominating graph of P_8 . By Lemma 2.1, we get $\Gamma_{pr}(P_8) = 6$. Then all $\Gamma_{pr}(P_8)$ -sets are $D_1 = \{v_1, v_2, v_3, v_4, v_6, v_7\}$, $D_2 = \{v_1, v_2, v_3, v_4, v_7, v_8\}$, $D_3 = \{v_1, v_2, v_4, v_5, v_6, v_7\}$, $D_4 = \{v_1, v_2, v_4, v_5, v_7, v_8\}$, $D_5 = \{v_1, v_2, v_5, v_6, v_7, v_8\}$, $D_6 = \{v_2, v_3, v_4, v_5, v_6, v_7\}$, $D_7 = \{v_2, v_3, v_4, v_5, v_7, v_8\}$, and $D_8 = \{v_2, v_3, v_5, v_6, v_7, v_8\}$. Then we have $PD_\Gamma(P_8) \cong SR(3,3)$.

Finally, we consider the Γ -paired dominating graph of P_{10} . By Lemma 2.1, we get $\Gamma_{pr}(P_{10}) = 8$. Then all $\Gamma_{pr}(P_{10})$ -sets are $D_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}$, $D_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\}$, $D_3 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9\}$, $D_4 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_9, v_{10}\}$, $D_5 = \{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}\}$, $D_6 = \{v_1, v_2, v_4, v_5, v_6, v_7, v_8, v_9\}$, $D_7 = \{v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_{10}\}$, $D_8 = \{v_1, v_2, v_4, v_5, v_7, v_8, v_9, v_{10}\}$, $D_9 = \{v_1, v_2, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $D_{10} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$, $D_{11} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}\}$, $D_{12} = \{v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}\}$, and $D_{13} = \{v_2, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}\}$.

Hence $PD_\Gamma(P_{10}) \cong SR(4,4)$ as shown in figure 6.

Fig. 6. The steprook $SR(4,4)$.

Let K_p be the complete graph with vertices $u_1, u_2, \dots, u_p$, K_q the complete graph with vertices $v_1, v_2, \dots, v_q$, and K_r the complete graph with vertices $w_1, w_2, \dots, w_r$. We define the steprook $SR(p, q, r)$ be the subgraph of $K_p \square K_q \square K_r$ induced by $\{(u_x, v_y, w_z) : x - y \leq 0, x - z \leq 1, y - z \geq 0\}$.

For instance, the steprooks $SR(2,2,1)$ and $SR(3,3,2)$ are shown in figures 7 and 8, respectively.

Fig.7. The steprook $SR(2,2,1)$.

Fig. 8. The steprook $SR(3,3,2)$.

Theorem 2.4. If $k \in \{3, 4, 5, 6\}$, then $PD_{\Gamma}(P_{2k+1}) \cong SR(k-1, k-1, k-2)$.

Proof. By Lemma 2.1, we get $\Gamma_{pr}(P_7) = 4$. Then the following are all $\Gamma_{pr}(P_7)$ -sets:
 $D_1 = \{v_1, v_2, v_5, v_6\}$, $D_2 = \{v_2, v_3, v_5, v_6\}$, and $D_3 = \{v_2, v_3, v_6, v_7\}$. Then we have $PD_{\Gamma}(P_7) \cong SR(2, 2, 1)$. If $k = 4$, then $\Gamma_{pr}(P_9) = 6$ and the following are all $\Gamma_{pr}(P_9)$ -sets:
 $D_1 = \{v_1, v_2, v_3, v_4, v_7, v_8\}$, $D_2 = \{v_1, v_2, v_4, v_5, v_7, v_8\}$, $D_3 = \{v_1, v_2, v_4, v_5, v_8, v_9\}$,
 $D_4 = \{v_1, v_2, v_5, v_6, v_7, v_8\}$, $D_5 = \{v_1, v_2, v_5, v_6, v_8, v_9\}$, $D_6 = \{v_2, v_3, v_4, v_5, v_7, v_8\}$,
 $D_7 = \{v_2, v_3, v_4, v_5, v_8, v_9\}$, $D_8 = \{v_2, v_3, v_5, v_6, v_7, v_8\}$, $D_9 = \{v_2, v_3, v_5, v_6, v_8, v_9\}$,
 and $D_{10} = \{v_2, v_3, v_6, v_7, v_8, v_9\}$.

Then we have $PD_{\Gamma}(P_9) \cong SR(3, 3, 2)$. If $k = 5$, then $\Gamma_{pr}(P_{11}) = 8$ and the following are all $\Gamma_{pr}(P_{11})$ -sets:

$D_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\}$, $D_2 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_9, v_{10}\}$,
 $D_3 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_{10}, v_{11}\}$, $D_4 = \{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}\}$,
 $D_5 = \{v_1, v_2, v_3, v_4, v_7, v_8, v_{10}, v_{11}\}$, $D_6 = \{v_1, v_2, v_4, v_5, v_6, v_7, v_9, v_{10}\}$,
 $D_7 = \{v_1, v_2, v_4, v_5, v_6, v_7, v_{10}, v_{11}\}$, $D_8 = \{v_1, v_2, v_4, v_5, v_7, v_8, v_9, v_{10}\}$,
 $D_9 = \{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}\}$, $D_{10} = \{v_1, v_2, v_4, v_5, v_8, v_9, v_{10}, v_{11}\}$,
 $D_{11} = \{v_1, v_2, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $D_{12} = \{v_1, v_2, v_5, v_6, v_7, v_8, v_{10}, v_{11}\}$,
 $D_{13} = \{v_1, v_2, v_5, v_6, v_8, v_9, v_{10}, v_{11}\}$, $D_{14} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}\}$,
 $D_{15} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_{10}, v_{11}\}$, $D_{16} = \{v_2, v_3, v_4, v_5, v_7, v_8, v_9, v_{10}\}$,
 $D_{17} = \{v_2, v_3, v_4, v_5, v_7, v_8, v_{10}, v_{11}\}$, $D_{18} = \{v_2, v_3, v_4, v_5, v_8, v_9, v_{10}, v_{11}\}$,
 $D_{19} = \{v_2, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $D_{20} = \{v_2, v_3, v_5, v_6, v_7, v_8, v_{10}, v_{11}\}$,
 $D_{21} = \{v_2, v_3, v_5, v_6, v_8, v_9, v_{10}, v_{11}\}$, and $D_{22} = \{v_2, v_3, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$.

Then we have $PD_{\Gamma}(P_{11}) \cong SR(4, 4, 3)$ as shown in figure 9.

Fig. 9. The steprook $SR(4,4,3)$.

Similarly, if $k = 6$, then $PD_{\Gamma}(P_{13}) \cong SR(5,5,4)$.

Finally, we consider the Γ -paired dominating graph of a path with 12 vertices. Then we obtain the graph shown in figure 10.

Fig. 10. The Γ -paired dominating graph of a path with 12 vertices.

3 Conclusion

As we show the Γ -paired dominating graphs of paths previously, we can notice that ones in the same set have similar structure:

$$\{PD_{\Gamma}(P_2)\}$$

$$\{PD_{\Gamma}(P_4),PD_{\Gamma}(P_6),PD_{\Gamma}(P_8),PD_{\Gamma}(P_{10})\}$$

$$\{PD_{\Gamma}(P_3),PD_{\Gamma}(P_5)\}$$

$$\{PD_{\Gamma}(P_7),PD_{\Gamma}(P_9),PD_{\Gamma}(P_{11}),PD_{\Gamma}(P_{13})\}$$

Let $n \geq 2$ be an integer. The upper paired domination numbers of P_n are shown in table 1.

Table 1. The upper paired domination numbers of paths

n	$\Gamma_{pr}(P_n)$	n	$\Gamma_{pr}(P_n)$
2	2	3	2
4	2	5	4
6	4	7	4
8	6	9	6
10	8	11	8
12	8	13	10
14	10	15	10
16	12	17	12
18	12	19	14
		21	16
		23	16

By considering the upper paired domination numbers, we think $PD_{\Gamma}(P_{14})$ and $PD_{\Gamma}(P_{16})$ should have similar structure as $PD_{\Gamma}(P_{12})$. Similarly, the Γ -paired dominating graphs of P_{15} , P_{17} , P_{19} , and P_{21} should have similar structure. Currently, we are working on the Γ -paired dominating graphs of paths with $n \geq 13$ vertices.

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