Some properties of the generalized (p,q)-Fibonacci-Like number

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Abstract. For the real world problems, we use some knowledge for explain or solving them. For example, some mathematicians study the basic concept of the generalized Fibonacci sequence and Lucas sequence which are the (p,q) – Fibonacci sequence and the (p,q) – Lucas sequence. Such as, Falcon and Plaza showed some results of the k-Fibonacci sequence. Then many researchers showed some results of the k-Fibonacci-Like number. Moreover, Suvarnamani and Tatong showed some results of the (p, q) - Fibonacci number. They found some properties of the (p,q) – Fibonacci number and the (p,q) – Lucas number. There are a lot of open problem about them. In this paper, we studied about the generalized (p,q)-Fibonacci-Like sequence. We establish properties like Catalan's identity, Cassini's identity, Simpson's identity, d'Ocagne's identity and Generating function for the generalized (p,q)-Fibonacci-Like number by using the Binet formulas. However, all results which be showed in this paper, are generalized of the (p,q) – Fibonacci-like number and the (p,q) – Fibonacci number. Corresponding author: kotmaster2@rmutt.ac.th

1 Introduction

Falcon and Plaza [1] showed some results of the k-Fibonacci sequence $\{F_{k,n}\}$ which is defined by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $F_{k,0} = 0, F_{k,1} = 1$. After that Falcon [2] found some properties of the k-Lucas sequence $\{L_{k,n}\}$ which is defined by $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $k \geq 1$ and $n \geq 1$ with $L_{k,0} = 2, L_{k,1} = k$. Then many researchers [3-5] showed some results of the k-Fibonacci-Like number in 2014.

In 2015, Suvarnamani and Tatong [6] proved some properties of the (p,q) - Fibonacci number which is defined by $F_{p,q,n+1} = pF_{p,q,n} + qF_{p,q,n-1}$ for $n \geq 1$ with $F_{p,q,0} = 0, F_{p,q,1} = 1$. Next, Suvarnamani [7] found some results of the (p,q)-Lucas number which is defined by $L_{p,q,n+1} = pL_{p,q,n} + qL_{p,q,n-1}$ for $n \geq 1$ with $L_{p,q,0} = 2, L_{p,q,1} = p$. Moreover, Suvarnamani showed more results of the (p,q)-Fibonacci number and the (p,q) - Lucas Number in [8-10].

In this paper, we will proved some identities of the (p,q)-Fibonacci-Like number and the generalized (p,q)-Fibonacci-Like number.

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2 Preliminaries

For p and q are positive real numbers, the (p,q) -Fibonacci-Like sequence $\{S_{p,q,n}\}$ is defined by $S_{p,q,n+1} = pS_{p,q,n} + qS_{p,q,n-1}$ for $n \in \mathbb{N}$ with $S_{p,q,0} = 2, S_{p,q,1} = 2p$. That is

$$\{S_{p,q,n}\} = \{2, 2p, 2p^2 + 2q, 2p^3 + 4pq, 2p^4 + 6p^2q + 2q^2, 2p^5 + 8p^3q + 6pq^2, \dots\}$$

Each term of the (p,q) -Fibonacci-Like sequence be called the (p,q) -Fibonacci-Like number. If $q=1$, we get the p -Fibonacci-Like sequence. That is

$$\{S_{p,1,n}\} = \{2, 2p, 2p^2 + 2, 2p^3 + 4p, 2p^4 + 6p^2 + 2, 2p^5 + 8p^3 + 6p, \dots\} = \{S_{p,n}\}.$$

For $p=q=1$, we get the Fibonacci-Like sequence. That is

$$\{S_{1,1,n}\} = \{2, 2, 4, 6, 10, 16, 26, \dots\} = \{S_n\}.$$

The Binet formulas of (p,q) -Fibonacci-Like sequence is given by

$$S_{p,q,n} = 2 \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2} \quad (1)$$

where R_1 and R_2 are roots of the characteristic equation $x^2 - px - q = 0$. So,

$$R_1 = \frac{p + \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad R_2 = \frac{p - \sqrt{p^2 + 4q}}{2}. \quad \text{Then we get } R_1 + R_2 = p, R_1 - R_2 = \sqrt{p^2 + 4q}, \\ R_1 R_2 = -q.$$

The generalized (p,q) -Fibonacci-Like sequence $\{T_{p,q,n}\}$ is defined by $T_{p,q,n+1} = pT_{p,q,n} + qT_{p,q,n-1}$ for $n \in \mathbb{N}$ with $T_{p,q,0} = m, T_{p,q,1} = mp, m \neq 0$. That is

$$\{T_{p,q,n}\} = \{m, mp, mp^2 + mq, mp^3 + 2mpq, mp^4 + 3mp^2q + mq^2, \dots\}.$$

Each term of the generalized (p,q) -Fibonacci-Like sequence be called the generalized (p,q) -Fibonacci-Like number. If $q=1$, we get the generalized p -Fibonacci-Like sequence. That is

$$\{T_{p,1,n}\} = \{m, mp, mp^2 + m, mp^3 + 2mp, mp^4 + 3mp^2 + m, \dots\} = \{T_{p,n}\}.$$

If $p=q=1$, we get the generalized Fibonacci-Like sequence. That is

$$\{T_{1,1,n}\} = \{m, m, 2m, 3m, 5m, 8m, 13m, \dots\} = \{T_n\}.$$

3 Main results

In this section, we present some of the interesting properties of the generalized (p,q) -Fibonacci-Like number like Catalan's identity, Cassini's identity, d'Ocagne's identity, Binet formulas and Generating function.

Theorem 3.1: (Binet formulas) If p and q are real numbers, then the n -th generalized (p,q) -Fibonacci-Like number $\{T_{p,q,n}\}$ is given by

$$T_{p,q,n} = m \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}. \quad (2)$$

Proof. Let $P(n): T_{p,q,n} = m \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}$ for $n \geq 0$. We use the principle of mathematical induction on n . We get $T_{p,q,1} = mp = m(R_1 + R_2) = m \frac{R_1^2 - R_2^2}{R_1 - R_2} = m \frac{R_1^{1+1} - R_2^{1+1}}{R_1 - R_2}$. Assume that it is true for r such that $0 \leq r \leq i+1$, then we have $T_{p,q,r} = m \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2}$. Then

$$\begin{aligned} T_{p,q,r+2} &= T_{p,q,(r+1)+1} \\ &= pT_{p,q,r+1} + qT_{p,q,r} \\ &= pm \frac{R_1^{r+2} - R_2^{r+2}}{R_1 - R_2} + qm \frac{R_1^{r+1} - R_2^{r+1}}{R_1 - R_2} \\ &= \frac{mR_1^{r+1}(pR_1 + q) - mR_2^{r+1}(pR_2 + q)}{R_1 - R_2} \\ &= \frac{mR_1^{r+1}(R_1^2) - mR_2^{r+1}(R_2^2)}{R_1 - R_2} \\ &= m \frac{R_1^{(r+2)+1} - R_2^{(r+2)+1}}{R_1 - R_2}. \end{aligned}$$

Thus, the formula is true for any positive integer n where $R_1 = \frac{p + \sqrt{p^2 + 4q}}{2}$ and $R_2 = \frac{p - \sqrt{p^2 + 4q}}{2}$. This completes the proof.

Corollary 3.2: If p and q are real numbers, then

$$T_{p,q,n} = \frac{m}{2} S_{p,q,n}. \quad (3)$$

Proof. From Theorem 3.1, we use formula (1), then we get

$$\begin{aligned} T_{p,q,n} &= m \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2} \\ &= \frac{m}{2} \left(2 \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2} \right) \\ &= \frac{m}{2} S_{p,q,n}. \end{aligned}$$

Thus, this completes the proof.

Theorem 3.3: (Catalan's identity) If p and q are real numbers, then

$$T_{p,q,n-r-1} T_{p,q,n+r-1} - T_{p,q,n-1}^2 = (-q)^{n-r} T_{p,q,r-1}^2. \quad (4)$$

Proof. By Theorem 3.1, we get

$$\begin{aligned}
 T_{p,q,n-r-1} T_{p,q,n+r-1} - T_{p,q,n-1}^2 &= \left(m \frac{R_1^{n-r} - R_2^{n-r}}{R_1 - R_2} \right) \left(m \frac{R_1^{n+r} - R_2^{n+r}}{R_1 - R_2} \right) - \left(m \frac{R_1^n - R_2^n}{R_1 - R_2} \right)^2 \\
 &= \frac{m^2 (R_1 R_2)^n}{(R_1 - R_2)^2} \left(- \left(\frac{R_2}{R_1} \right)^r - \left(\frac{R_1}{R_2} \right)^r + 2 \right) \\
 &= (-q)^{n-r} m^2 \frac{(R_1^r - R_2^r)^2}{(R_1 - R_2)^2} \\
 &= (-q)^{n-r} T_{k,r-1}^2.
 \end{aligned}$$

Thus, this completes the proof.

Theorem 3.4: (Catalan's identity or Simpson's identity) If p and q are real numbers, then

$$T_{p,q,n-2} T_{p,q,n} - T_{p,q,n-1}^2 = (-q)^{n-1} m^2. \quad (5)$$

Proof. From Theorem 3.1, if $r = 1$, we get

$$\begin{aligned}
 T_{p,q,n-2} T_{p,q,n} - T_{p,q,n-1}^2 &= T_{p,q,n-1-1} T_{p,q,n+1-1} - T_{p,q,n-1}^2 \\
 &= (-q)^{n-1} T_{p,q,1-1}^2 \\
 &= (-q)^{n-1} T_{p,q,0}^2 \\
 &= (-q)^{n-1} m^2.
 \end{aligned}$$

Thus, this completes the proof.

Theorem 3.5: (d'Ocagne's identity) If p and q are real numbers, then

$$T_{p,q,m-1} T_{p,q,n} - T_{p,q,m} T_{p,q,n-1} = (-1)^n m T_{p,q,m-n-1}. \quad (6)$$

Proof. By From Theorem 3.1, we get

$$\begin{aligned}
 T_{p,q,m-1} T_{p,q,n} - T_{p,q,m} T_{p,q,n-1} &= \left(m \frac{R_1^m - R_2^m}{R_1 - R_2} \right) \left(m \frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2} \right) - \left(m \frac{R_1^{m+1} - R_2^{m+1}}{R_1 - R_2} \right) \left(m \frac{R_1^n - R_2^n}{R_1 - R_2} \right) \\
 &= \frac{m^2}{R_1 - R_2} (R_1^m R_2^n - R_1^n R_2^m) \\
 &= \frac{m^2 R_1^n R_2^n}{R_1 - R_2} (R_1^{m-n} - R_2^{m-n}) \\
 &= (-q)^n m T_{p,q,m-n-1}.
 \end{aligned}$$

Thus, this completes the proof.

Theorem 3.6: If p and q are real numbers, then

$$\lim_{n \rightarrow \infty} \left(\frac{T_{p,q,n-1}}{T_{p,q,n-2}} \right) = R_1 \quad (7)$$

Proof. By From Theorem 3.1, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{T_{p,q,n-1}}{T_{p,q,n-2}} \right) &= \lim_{n \rightarrow \infty} \frac{R_1^n - R_2^n}{R_1^{n-1} - R_2^{n-1}} \\ &= \lim_{n \rightarrow \infty} \frac{1 - \left(\frac{R_2}{R_1} \right)^n}{\frac{1}{R_1} - \left(\frac{R_2}{R_1} \right)^n \frac{1}{R_2}} \\ &= R_1. \end{aligned}$$

Thus, this completes the proof.

In this paper, the generating function for the generalized (p,q)-Fibonacci-Like sequence is given. As a result, the generalized (p,q)-Fibonacci-Like sequence is seen and the coefficient of the power series of the corresponding generating function. Let us suppose that the generalized (p,q)-Fibonacci-Like number of order p is the coefficient of a potential series center at the origin, and let us consider the corresponding analytic that the function $\{T_{p,q,n}\}$ defined in such a way is called the generating function of the generalized (p,q)-Fibonacci-Like number. So, $T_{p,q}(x) = T_{p,q,0} + T_{p,q,1}x + T_{p,q,2}x^2 + \dots + T_{p,q,n}x^n$.

Then $(q - px - x^2)T_{p,q}(x) = qT_{p,q}(x) - pxT_{p,q}(x) - x^2T_{p,q}(x) = m + mqx - mpq$.

Thus $T_{p,q}(x) = \frac{m + mqx - mpq}{q - px - x^2}$.

4 Conclusion

In this paper, the generalized (p,q)-Fibonacci-Like sequence have been introduced and studied. The properties of number are proved by Binet formulas. We obtain properties like Catalan's identity, Cassini's identity, Simpson's identity and d'Ocagne's identity for the generalized (p,q)-Fibonacci-Like number.

Acknowledgment

This research was partly supported by Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathum Thani, THAILAND. (RMUTT Annual Government Statement of Expenditure in 2017)

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