

An elementary proof that real roots nonexistence of nonlinear equations and its optimal solutions

Lanyou Li^{1,2}, Jingui Lu^{1,a} and Jianhong Chen²

¹School of Mechanical and Power Engineering, Nanjing Tech University, Nanjing Jiangsu 211800,China

²Office of Information Construction and Management, Nanjing Institute of Technology, Nanjing Jiangsu 211167,China

Abstract. In this paper, we give an elementary proof that the real roots nonexistence of nonlinear equation such as: $x^n+y^n+z^n=n$, where $n=1,2,3, (x,y,z) \in \mathbb{R}$ by contradiction. Thus, when n takes all positive integers between 4 and 1981, the value of function $f=x^n+y^n+z^n$ based on the exact solution of the above equations is still real. Finally, numerical experiments presents a set of approximate numerical solutions by using standard genetic algorithm(SGA), adaptive particle swarm optimization algorithm(APSO) and artificial fish swarm algorithm(AFSA).

1 Introduction

We consider the nonlinear equations given by equations group (1) with the condition x,y and z belongs to \mathbb{R} .

$$\begin{cases} x + y + z = 1 \\ x^2 + y^2 + z^2 = 2 \\ x^3 + y^3 + z^3 = 3 \end{cases} \quad (1)$$

Thus we can prove that the Eq.(1) has no real roots[1] and discuss the value of the function such as:

$$f = x^n + y^n + z^n, \quad (2)$$

which is based on the exact solution of Eq. (1) and $n \in \mathbb{Z}^+$. The main result is stated as follows:

Theorem 1. *Let x,y and z be real numbers, the Eq.(1) has no solutions in real number field.*

In the following section, we will prove Theorem 1 by contradiction[2,3]. We first assume that a solution, (x,y,z) , to Eq.(1) exists, and then prove that it yields a contradiction. The paper is organized as follows: In Section 2, we give a completely elementary proof of the nonexistence result of Eq.(1) in Theorem 1. The numerical experiments and some bionic inspired algorithms are presented in Section 3. Section 4 summarizes the conclusions and points deserving further attention.

2 Proof of Theorem 1

Let x,y , and z , as in Theorem 1, be given, and suppose that $x,y,z \in \mathbb{R}$ satisfy Eq.(1). Thus, there exist three equalities such that

$$x + y + z = 1, \quad (3)$$

$$x^2 + y^2 + z^2 = 2, \quad (4)$$

$$x^3 + y^3 + z^3 = 3. \quad (5)$$

Using equality (3), we obtain

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz) = 1. \quad (6)$$

Substituting equality (4) into equality (6), we have that

$$xy + xz + yz = -1/2. \quad (7)$$

Obviously, we also have that

$$x^3 + y^3 + z^3 - 3xyz \quad (8)$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Therefore, combining equalities (3)-(5), (7) and (8), we obtain

$$xyz = 1/6. \quad (9)$$

Thus, we find that

$$x^2y^2 + x^2z^2 + y^2z^2 \quad (10)$$

$$= (xy + xz + yz)^2 - 2xyz(x + y + z).$$

Substituting equalities (7), (9), and (3) into equality (10), we have

$$x^2y^2 + x^2z^2 + y^2z^2 = -1/12. \quad (11)$$

Hence, by equality (11), this implies that the inequality such as $(xy)^2 + (xz)^2 + (yz)^2 < 0$ holds. It can be easily verified that this yields a contradiction clearly[4]. Thus, there is no such solution to Eq.(1), so this accomplishes the proof.

Remark 1. *Another method of solving the equality (9) in the above proof process is given as follows:*

From equality (3) and (4) it follows that

$$\begin{aligned} (x + y + z)(x^2 + y^2 + z^2) \\ = (x^3 + y^3 + z^3) + (x+y)(xy + z^2) + z(x^2 + y^2) = 2. \end{aligned} \quad (12)$$

Substituting equality (5) into equality (12), we have that

$$(x+y)(xy + z^2) + z(x^2 + y^2) = -1. \quad (13)$$

*Corresponding author: ^a lujingui@163.com, lilanyou@163.com

We find that

$$x^3 + y^3 + z^3 = (x + y + z)^3 - 3[(x+y)(xy + z^2) + z(x^2 + y^2) + 2xyz]. \quad (14)$$

Combining equalities (3),(13),(5) and (14),we obtain the equality (9).

3 Numerical experiments

3.1 Derivation of elementary methods

In this section, we can obtain the value of function (2) based on the exact solution of Eq.(1) using elementary methods such as: when $n=4$, the value of function (2) is equal to $25/6$, when $n=5$,the value of function (2) is equal to 6 and when $n=6$, the value of function (2) is equal to $103/12$.

Using equality (4),we obtain

$$(x^2 + y^2 + z^2)^2 = x^4 + y^4 + z^4 + 2(x^2y^2 + x^2z^2 + y^2z^2) = 4. \quad (15)$$

Combining equalities (11) and (15),we obtain

$$x^4 + y^4 + z^4 = 25 / 6. \quad (16)$$

So the value of function (2) is shown with the parameter $n=4$.

Using equalities (3) and (16),we obtain

$$(x + y + z)(x^4 + y^4 + z^4) = 25 / 6. \quad (17)$$

The equality (5) can be rewritten as

$$x^3 + y^3 = 3 - z^3, \quad (18)$$

$$x^3 + z^3 = 3 - y^3, \quad (19)$$

$$y^3 + z^3 = 3 - x^3, \quad (20)$$

Using equality (17),it gives

$$x^5 + y^5 + z^5 + xy(x^3 + y^3) + xz(x^3 + z^3) + yz(y^3 + z^3) = 25 / 6. \quad (21)$$

Combining equalities (18),(19),(20) and (21),we obtain

$$x^5 + y^5 + z^5 + xy(3 - z^3) + xz(3 - y^3) + yz(3 - x^3) = x^5 + y^5 + z^5 + 3(xy + xz + yz) - xyz(x^2 + y^2 + z^2), \text{ it}$$

yields

$$x^5 + y^5 + z^5 + 3(xy + xz + yz) - xyz(x^2 + y^2 + z^2) = 25 / 6. \quad (22)$$

Substituting equalities (7), (9), and (4) into equality (22),we have

$$x^5 + y^5 + z^5 = 6. \quad (23)$$

So the value of function (2) is shown with the parameter $n=5$.

Using equality (7) and (11),we obtain

$$(xy + xz + yz)(x^2y^2 + x^2z^2 + y^2z^2) = 1 / 24. \quad (24)$$

The equality (24) can be rewritten as

$$(x^3y^3 + x^3z^3 + y^3z^3) + xyz[z(x^2 + y^2) + y(x^2 + z^2) + x(y^2 + z^2)] = 1 / 24. \quad (25)$$

The equality (4) can be rewritten as

$$x^2 + y^2 = 2 - z^2, \quad (26)$$

$$x^2 + z^2 = 2 - y^2, \quad (27)$$

$$y^2 + z^2 = 2 - x^2. \quad (28)$$

Combining equalities (9),(26),(27),(28) and (25),we obtain

$$(x^3y^3 + x^3z^3 + y^3z^3) + [2(x + y + z) - (x^3 + y^3 + z^3)] / 6 = 1 / 24. \quad (29)$$

Substituting equalities (3) and (5) into the equality (29),we have

$$x^3y^3 + x^3z^3 + y^3z^3 = 5 / 24. \quad (30)$$

Using equality (5),we obtain

$$(x^3 + y^3 + z^3)^2 = 9. \quad (31)$$

The equality (31) can be rewritten as

$$x^6 + y^6 + z^6 + 2(x^3y^3 + x^3z^3 + y^3z^3) = 9. \quad (32)$$

Substituting equality (30) into the equality (32),we have

$$x^6 + y^6 + z^6 = 103 / 12. \quad (33)$$

So the value of function (2) is shown with the parameter $n=6$.What follows is using Matlab's solve function to obtain the exact solutions for a calculation when n takes all positive integers between 4 and 1981:

$S = \text{solve}('x+y+z=1', 'x^2+y^2+z^2=2', 'x^3+y^3+z^3=3', 'x', 'y', 'z', 'MaxDegree', 4)$

Then, we obtain six groups complex roots of Eq.(1). Using these, we can get the value of function (2).Table 1 illustrates the value of function(2) when n can be given from 1 to 30. When n is equal to 1981,the value of function(2) is approximately 1.704774E308+0i.Numeric experiment shows that the value of function (2) is still real until n is equal to 1981.

Table 1.The value of function(2) when n given from 1 to 30.

n	$f=x^p+y^p+z^p$	n	$f=x^p+y^p+z^p$	n	$f=x^p+y^p+z^p$
1	1.0000e0+0i	11	5.1468e1+0i	21	1.8513e3+0i
2	2.0000e0+0i	12	7.3642e1+0i	22	2.6489e3+0i
3	3.0000e0+0i	13	1.0537e2+0i	23	3.7902e3+0i
4	4.1667e0+0i	14	1.5077e2+0i	24	5.4232e3+0i
5	6.0000e0+0i	15	2.1573e2+0i	25	7.7598e3+0i
6	8.5833e0+0i	16	3.0868e2+0i	26	1.1103e4+0i
7	1.2278e1+0i	17	4.4167e2+0i	27	1.5887e4+0i
8	1.7569e1+0i	18	6.3196e2+0i	28	2.2732e4+0i
9	2.5139e1+0i	19	9.0424e2+0i	29	3.2526e4+0i
10	3.5970e1+0i	20	1.2938e3+0i	30	4.6539e4+0i

3.2 The optimal approximate real solutions

Consider the general forms of nonlinear equations

$$\left\{ \begin{array}{l} f_1(x_1, x_2, \dots, x_n) = A_1 \\ f_2(x_1, x_2, \dots, x_n) = A_2 \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = A_n \end{array} \right\}. \quad (34)$$

where $f_i(x_1, x_2, \dots, x_n) = A_i (i=1, 2, \dots, n)$ is a real-valued function on area D of the n -dimensional Euclidean space \mathbb{R} , and at least one of which is a nonlinear function, $X=(x_1, x_2, \dots, x_n)^T$ is the vector variables and $A_i (i=1, 2, \dots, n)$ is the constant. In order to apply intelligent optimization algorithms for solving nonlinear equations, the problem of solving nonlinear equations is transformed into a nonlinear least squares problem, formulated as a function optimization problem:

$$\left\{ \begin{array}{l} \text{find} : X = [x_1, x_2, \dots, x_n], X \in \Phi \\ \text{min} : F(X) = \sum_{i=1}^n [f_i(x) - A_i]^2 \end{array} \right\}. \quad (35)$$

Where Φ is the solution space of Eq.(34). So when $F(X)$ takes the minimum value 0, the corresponding solution $X^* = \{x_1^*, x_2^*, \dots, x_n^*\}$ of (35) is the solution of Eq.(34). For Eq.(1), it is impossible to find a set of exact real number solutions such that $F(X)=0$, but we can find a set of approximate numerical solutions to make sure the value of $F(X)$ is minimized. We make use of standard genetic algorithm(SGA)[5,6], adaptive particle swarm optimization algorithm(PSO)[7] and artificial fish swarm algorithm(AFSA) for solving the equations. Artificial fish swarm algorithm was presented by Li et al.[8,9] and improved in Yazdani et al.[10] and He et al.[11]. It is inspired from the fish swarm behaviors which contains preying behavior, swarming behaviour and following behavior[12]. In general, it works as follows[13]:

Step 1: Initialize: the population size is $fishnum$, the biggest evolution generation is $maxgen$, the visual distance is $visual$, the crowded factor is $\delta (0 < \delta < 1)$, the largest trying number is try_number and so on. Initialize the current position $X[fishnum]$ for $fishnum$ artificial fishes randomly. Calculate the food consistence

$Y[fishnum]$ for every artificial fishes. Mark the bulletin board.

Step 2: Each artificial fish in the population performs swarming behavior and following behavior with preying behavior if needed, to search for the position where the food resources are richer than the current position. If the position after finishing an iteration is better than the bulletin board, update bulletin board and go to step 3.

Step 3: Evolution generation $gen=gen+1$. If $gen < maxgen$, go to step 2. Otherwise go to step 4.

Step 4: The algorithm reaches an end. Output the best optimal solutions.

Herein we present the standard genetic algorithm that is implemented within a ga function from *MATLAB*TM Optimization Toolbox to perform the tests. As described in [7], the adaptive particle swarm optimization algorithm is given. Next, we now consider the fitness function such as:

$$Y = (x^2 + y^2 + (1 - x - y)^2 - 2)^2 + (x^3 + y^3 + (1 - x - y)^3 - 3)^2. \quad (36)$$

The parameters for SGA are as follows: the population size $PopulationSize$ is equal to 300, the maximum iteration $Generations$ is equal to 150, the tolerance $TolFun$ is equal to $1e-10$.

The parameters for APSO algorithm are as follows: the population size $PopulationSize$ is equal to 300, the maximum iteration $Generations$ is equal to 150, the maximum $weight$ is equal to 0.9, the minimum $weight$ is equal to 0.6.

The parameters for AFSA algorithm are as follows: $fishnum=300, maxgen=150, visual=0.5, step = 0.01, \delta = 0.618, try_number=100$.

The evolution implementation process of above three algorithms is given in Fig.1. The optimal coordinate movement trajectory and the fitness function image with AFSA algorithm is shown in Fig.2.

Table 2. The optimal approximate numerical solutions according to fitness functions (36)

algorithms	x	y	z	F
SGA	-2.2750e-01	-2.0478e-01	1.4323e00	2.7807e-02
APSO	-2.1630e-01	-2.1630e-01	1.4326e00	2.7699e-02
AFSA	-2.1631e-01	-2.1630e-01	1.4326e00	2.7699e-02

Table 2 illustrates the optimal approximate numerical solutions according to the fitness functions (36). Then we can see that the optimal approximate numerical solutions

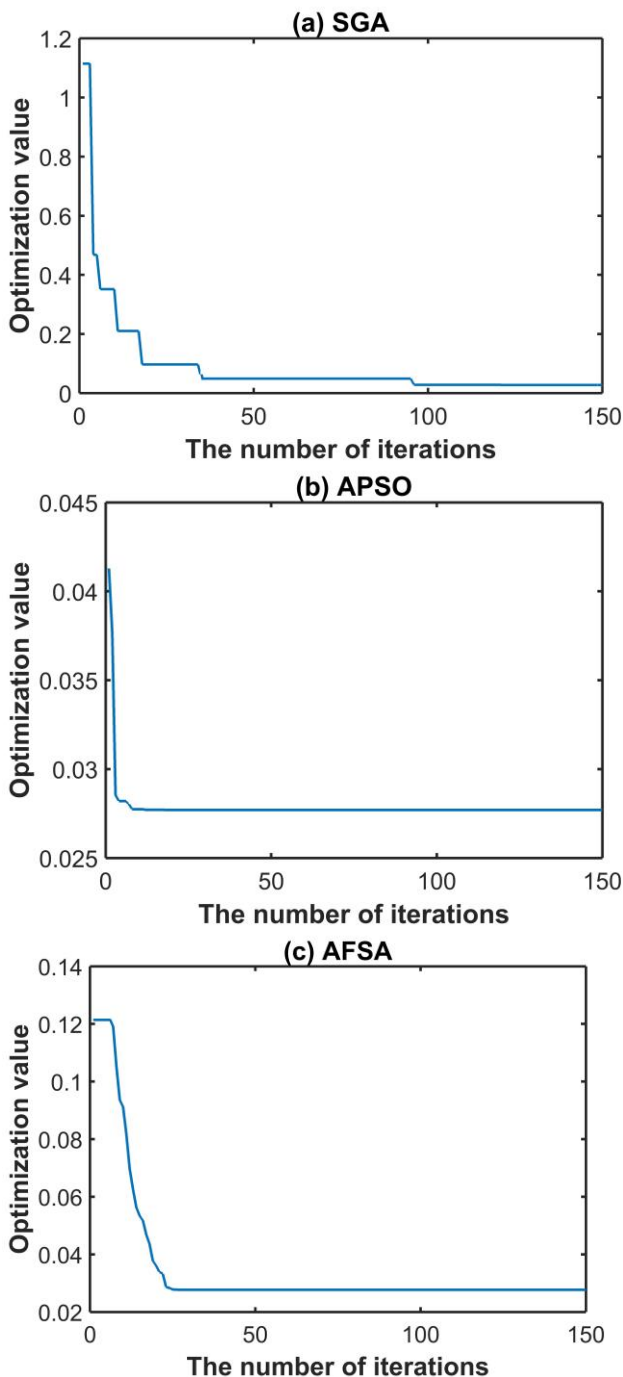


Figure 1. Optimization iterative process with function (36) and different algorithms such as:SGA,APSO and AFSA.

is $(-2.1631e-01, -2.1630e-01, 1.4326e00)$. All results are computed by *LenovoTM* ThinkServer RD450 with Intel Xeon CPU E5-2630 v3@2.40GHz,128GB RAM under Windows Server 2008R2 Enterprise platform and programs are coded in *MATLABTM* R2015a.

4 Conclusion and future work

In this note, we give a proof that is completely elementary to establish the real roots nonexistence of Eq.(1). Even more importantly, the value of the function(2) when n takes all positive integers between 4 and 1981 is still real, which is calculated in numeric

experiment. We conjecture that when n takes all positive integers, the value of function (2) may be always a real number. To the best of our knowledge, however, no strict mathematical proof method has been found. It is expected that the strict mathematical proof method is shown in future work.

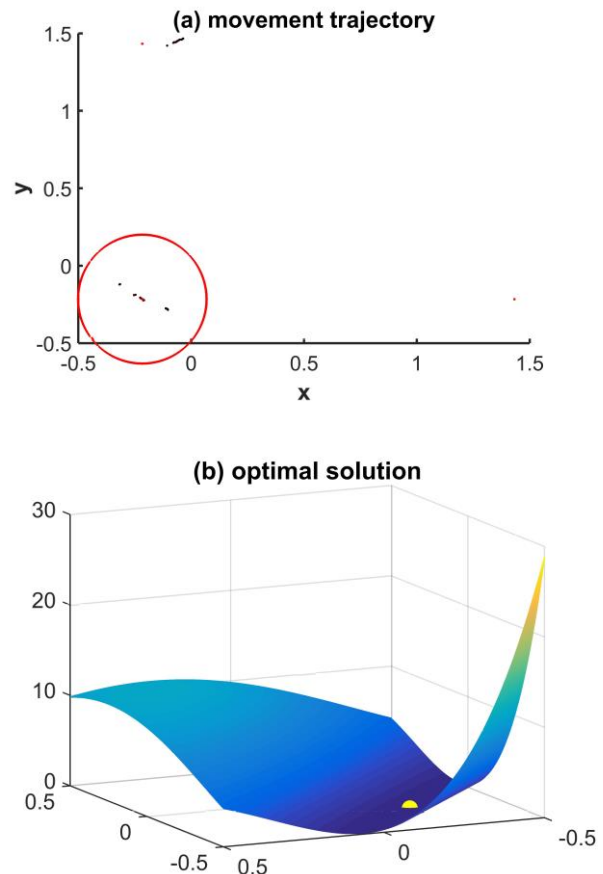


Figure 2.(a)the movement trajectory of fish swarm,(b)the fitness image and iterative optimal solution.

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