Study on the theory of three-dimensional limit analysis of soil and practical calculation

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Abstract. Based on the stress analysis of a point in space, the stress analysis of a point on a surface is performed, and then the relationship among the normal stress, the shear stress, and the stress component of a point on a surface and the first order partial derivative of the surface equation is deduced. Equilibrium equations of soil columns between the sliding surface and the top surface of the slope are established, which include differential equilibrium equation of force, equilibrium equation of force, and equilibrium equation of moment. These equilibrium equations and Coulomb yield condition can form the fundamental equations of three-dimensional slope stability analysis. Applying the supposition similar to that applied in the simplified Bishop method, a kind of three-dimensional slope analysis method can be obtained. An example is presented to show that the computation method is reasonable and applicable.

1 INTRODUCTION

The bearing capacity of the foundation, the stability of the slope and the earth pressure are the main problem in the engineering practice. For the problem of slope stability, no matter artificial slope or natural slope, the situation of landslide occurs is under three dimensional conditions, so the engineering meaning of 3D slope stability analysis is beyond doubt. Although the three-dimensional stability analysis is of great significance, the related stability analysis methods and program development work are not meeting the actual needs. Most of the research work is limited to academic field. The existing 3D analysis methods have a lot of assumptions.

The problem of the bearing capacity of foundation is the shape of the foundation in engineering practice, the strip foundation (2D) is not exist, although in many cases can be simplified as a two-dimensional problem, or by two-dimensional approximate calculation; but it should be considered as 3D problem.

From the theoretical point of view, the problem of yield criterion is the basis of the theory of three-dimensional limit analysis. However, up to now, there is no uniform definition for the stress field of the soil in the limit state. The fact is that no one type yield criterion can prove that the stress state is the most likely to cause damage of the soil.

There is no clear definition of the sliding surface. For example, the base yield criterion is a space stress

analysis, and no connection with the space surface; in fact, the limit state of stress field is connected with yield surface or sliding surface. The limit state of the soil is only along with the sliding surface, and the other surface is in a stable state (or limit state).

In addition, the existing yield criterion is only one of the equations that should be satisfied for the limit state of soil. It cannot form a complete set of basic equations together with the equilibrium equation. Under these conditions, only assumed conditions or other methods (constitutive models) can be used to determine the stress field.

It can be said that the research on the theory and application of three-dimensional soil limit analysis is behind engineering practice, and it isn't fully meeting the engineering needs.

2 The establishment of the equation set

2.1 Research on the one point stress analysis in space

(1) The basic relation of a point of stress in space

$$\left. \begin{aligned} \sigma_x l + \tau_{xy} m + \tau_{xz} n &= \sigma_t + \sigma l \\ \tau_{xy} l + \sigma_y m + \tau_{yz} n &= \tau m_t + \sigma m \\ \tau_{xz} l + \tau_{yz} m + \sigma_z n &= \tau n_t + \sigma n \end{aligned} \right\} \quad (1)$$

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$$\begin{aligned} \tau^2 + (\sigma - \sigma_m)^2 &= p_t^2, \\ p_t^2 &= [(\sigma_x - \sigma_m)l + \tau_{xy}m + \tau_{xz}n]^2 \\ &+ [\tau_{xy}l + (\sigma_y - \sigma_m)m + \tau_{yz}n]^2 \\ &+ [\tau_{xz}l + \tau_{yz}m + (\sigma_z - \sigma_m)n]^2 \end{aligned} \quad (2)$$

$$\sigma_y + \frac{c}{\lambda} = (\sigma + \frac{c}{\lambda}) [1 + \lambda^2 (\frac{1+h_x'^2}{1+h_x'^2+h_y'^2} + \frac{\sin^2 \alpha}{1+A_\alpha^2}) - \frac{2\lambda h_y' \sin \alpha}{\sqrt{1+h_x'^2+h_y'^2} \sqrt{1+A_\alpha^2}}] \quad (5b)$$

$$\sigma_z + \frac{c}{\lambda} = (\sigma + \frac{c}{\lambda}) [1 + \lambda^2 (\frac{h_x'^2+h_y'^2}{1+h_x'^2+h_y'^2} + \frac{A_\alpha^2}{1+A_\alpha^2}) + \frac{2\lambda A_\alpha}{\sqrt{1+h_x'^2+h_y'^2} \sqrt{1+A_\alpha^2}}] \quad (5c)$$

$$\tau_{xy} = -\lambda(\sigma + \frac{c}{\lambda}) [\lambda(\frac{h_x' h_y'}{1+h_x'^2+h_y'^2} - \frac{\cos \alpha \sin \alpha}{1+A_\alpha^2}) + \frac{h_y' \cos \alpha + h_x' \sin \alpha}{\sqrt{1+h_x'^2+h_y'^2} \sqrt{1+A_\alpha^2}}] \quad (5d)$$

$$\tau_{xz} = \lambda(\sigma + \frac{c}{\lambda}) [\lambda(\frac{h_x'}{1+h_x'^2+h_y'^2} + \frac{A_\alpha \cos \alpha}{1+A_\alpha^2}) + \frac{\cos \alpha - h_x' A_\alpha}{\sqrt{1+h_x'^2+h_y'^2} \sqrt{1+A_\alpha^2}}] \quad (5e)$$

$$\tau_{yz} = \lambda(\sigma + \frac{c}{\lambda}) [\lambda(\frac{h_y'}{1+h_x'^2+h_y'^2} + \frac{A_\alpha \sin \alpha}{1+A_\alpha^2}) + \frac{\sin \alpha - h_y' A_\alpha}{\sqrt{1+h_x'^2+h_y'^2} \sqrt{1+A_\alpha^2}}] \quad (5f)$$

$$A_\alpha = h_x' \cos \alpha + h_y' \sin \alpha$$

α is the direction angle of the shear stress on the horizontal plane. Formula (5) is the limit stress field criterion.

The ultimate stress criterion and the equilibrium equations constitute a complete set of basic equations, lays theoretical basis for 3D limit analysis, limit analysis method of this theory fully complies with the limit state design principles, the complete set of equations is also first proposed.

3 analysis of three dimensional slope

The limit stress field criterion is introduced into the equilibrium equation, and the limit equilibrium equation is obtained. The equations have three unknown quantities: normal stress, yield surface and horizontal direction of failure. The derivation process is too complicated to explain in detail.

The equation along the failure surface is as follows:

$$\begin{aligned} & [\frac{\partial}{\partial x}(\sigma + \frac{c}{\lambda}) + h_x' \frac{\partial}{\partial z}(\sigma + \frac{c}{\lambda})] C_\alpha + [\frac{\partial}{\partial y}(\sigma + \frac{c}{\lambda}) + h_y' \frac{\partial}{\partial z}(\sigma + \frac{c}{\lambda})] S_\alpha \\ & - \lambda(\sigma + \frac{c}{\lambda}) (\frac{\partial h_x'}{\partial x} + h_x' \frac{\partial h_y'}{\partial z}) \frac{1+h_x'^2+C_\alpha^2}{1+h_x'^2+h_y'^2} + \lambda(\sigma + \frac{c}{\lambda}) (\frac{\partial h_y'}{\partial y} + h_y' \frac{\partial h_x'}{\partial z}) \frac{h_x' h_y' - S_\alpha C_\alpha}{1+h_x'^2+h_y'^2} \\ & + \lambda(\sigma + \frac{c}{\lambda}) (\frac{\partial h_x'}{\partial x} + h_x' \frac{\partial h_y'}{\partial z}) \frac{h_x' h_y' - S_\alpha C_\alpha}{1+h_x'^2+h_y'^2} - \lambda(\sigma + \frac{c}{\lambda}) (\frac{\partial h_y'}{\partial y} + h_y' \frac{\partial h_x'}{\partial z}) \frac{1+h_x'^2+S_\alpha^2}{1+h_x'^2+h_y'^2} \\ & = \gamma \frac{1}{1+\lambda^2} (h_x' C_\alpha + h_y' S_\alpha - \lambda) \end{aligned} \quad (6a)$$

If the area enclosed by the surface $z = h(x, y)$ and the surface of the soil is closed area: Ω , the virtual work equation is as follow:

$$\begin{aligned} & \iint_D (-p_x v_{xs} - p_y v_{ys} - p_z v_{zs}) dx dy = \iiint_\Omega w_i dz dx dy \\ & - \iint_D \frac{c}{\lambda} [h_x' v_{xs} + h_y' v_{ys} - v_{zs}]_{z=h} dx dy - \iiint_\Omega (\frac{\partial v_{xs}}{\partial x} + \frac{\partial v_{ys}}{\partial y} + \frac{\partial v_{zs}}{\partial z}) \frac{c}{\lambda} dz dx dy \end{aligned} \quad (6b)$$

The imaginary work equation contains only the velocity component and the yield surface (or the sliding surface) without stress.

$$\begin{aligned} & C_\alpha \frac{\partial v}{\partial x} + S_\alpha \frac{\partial v}{\partial y} + (h_x' C_\alpha + h_y' S_\alpha) \frac{\partial v}{\partial z} + v [\frac{\partial C_\alpha}{\partial x} + \frac{\partial S_\alpha}{\partial y} + \frac{\partial}{\partial z} (h_x' C_\alpha + h_y' S_\alpha)] \\ & + v \lambda [(\frac{\partial h_x'}{\partial x} + h_x' \frac{\partial h_y'}{\partial z}) \frac{1+h_x'^2+C_\alpha^2}{1+h_x'^2+h_y'^2} - (\frac{\partial h_y'}{\partial x} + h_y' \frac{\partial h_x'}{\partial z}) \frac{h_x' h_y' - S_\alpha C_\alpha}{1+h_x'^2+h_y'^2} \\ & - (\frac{\partial h_x'}{\partial y} + h_y' \frac{\partial h_x'}{\partial z}) \frac{h_x' h_y' - S_\alpha C_\alpha}{1+h_x'^2+h_y'^2} + (\frac{\partial h_y'}{\partial y} + h_y' \frac{\partial h_y'}{\partial z}) \frac{1+h_x'^2+S_\alpha^2}{1+h_x'^2+h_y'^2}] = 0 \end{aligned}$$

(6c)

Formula (6c) is the velocity equation.

This is the first time to analyse the stability of the three dimensional slope under a hypothetical condition. The similar method has got according to the 2D slope simplified bishops method.

$$M_R = \iint_D \frac{(w_\gamma + p_z - u) \tan \phi + c}{\frac{\Delta_\alpha}{\Delta_h} + \lambda_F (h_x' \cos \alpha + h_y' \sin \alpha)}$$

$$\{[A_1(h-z_R) + A_3(y-y_R)][\cos \alpha + h_x' (h_x' \cos \alpha + h_y' \sin \alpha)] + [A_2(h-z_R) - A_3(x-x_R)][\sin \alpha + h_y' (h_x' \cos \alpha + h_y' \sin \alpha)]\} dx dy \quad (6d)$$

$$\begin{aligned} M_0 &= \iint_D (w_\gamma + p_z) \{h_x' [A_1(h-z_R) + A_3(y-y_R)] \\ &+ h_y' [A_2(h-z_R) - A_3(x-x_R)]\} dx dy \\ &+ \iint_D \{p_x [A_1(s-z_R) + A_3(y-y_R)] \\ &+ p_y [A_2(s-z_R) - A_3(x-x_R)]\} dx dy \end{aligned} \quad (6e)$$

The safety factor is still calculated by $F_s = M_R / M_0$.

a local load is on the 2D slope, which is a common case in engineering, It should be calculated in a three-dimensional way.

Figure 2. Diagram of local loading on Slope

Table 1. The safety factor of two-dimensional slope

sliding surface ^{a)}	condition ^{a)}	$p_z = 30\text{kPa}^a)$		$p_z = 60\text{kPa}^a)$	
		Width of load ^{a)}		width of load ^{a)}	
		15m ^{a)}	30m ^{a)}	15m ^{a)}	30m ^{a)}
Ellipsoidal surface ^{a)}	$\varphi = 20^\circ$ ^{a)}	1.464 ^{a)}	1.457 ^{a)}	1.350 ^{a)}	1.431 ^{a)}
	$\varphi = 25^\circ$ ^{a)}	1.722 ^{a)}	1.714 ^{a)}	1.713 ^{a)}	1.530 ^{a)}
Ellipsoidal-cylindrical-ellipsoidal surface ^{a)}	$\varphi = 20^\circ$ ^{a)}	1.460 ^{a)}	1.444 ^{a)}	1.350 ^{a)}	1.428 ^{a)}
	$\varphi = 25^\circ$ ^{a)}	1.715 ^{a)}	1.700 ^{a)}	1.705 ^{a)}	1.530 ^{a)}

Numerical examples show that the calculation results of this method are reasonable, This method is incomparable than other three dimensional slope stability analysis method who has a lot of assumptions.

4 Conclusions

(1) The velocity equation along with the sliding surface is proposed for the first time, which obtained stress equation, differential column force balance equation, torque balance equation, virtual work equation; these equations are universally applicable equation in three-dimensional limit analysis. It provides a theoretical basis for the calculation of three-dimensional foundation bearing capacity and the slope stability.

(2) A variety of methods for solving three-dimensional limit analysis are proposed. These methods are similar to two dimensional problem solving methods. That is to say, a scientific research method for three-dimensional slope stability and foundation bearing capacity is found.

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