An Optimization Method for Design of Acoustic Metamaterial Structure

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Abstract. Pentamode (PM) metamaterial is a kind of acoustic metamaterial generally designed from a general material and made by a periodic array of micro-truss structures. The paper presents an optimization method for the design of PM metamaterial structure, and this kind of structure has usually special physical properties to guide the acoustic wave to propagate according to the design path. In order to construct the optimization model, the micro-truss unit cell is firstly investigated deeply, and the relationship between the effective elastic modulus of PM materials and the structural parameters of micro-truss unit cells is established. With incorporating the transformation acoustics algorithm into the optimization process to predict the theoretic material parameters of the current design structure, the constraint of the material property in the design structure is converted into the rod size constraint of micro-truss. As a result, the design of PM material structures can be realized as a problem of structure size optimization, and the optimized result can meet the requirement of PM metamaterial property and the theory of transformation acoustics. And also, the total stability of the PM material structure is also ensured by the balance rod forces constraints in the optimization process. Finally, a numerical example of PM material structure is presented.

1 Introduction

Over the past few decades, people have developed interest in the study of superior properties and potential applications of optical and electromagnetic metamaterials. Recently, the attention has been extended to acoustic and elastic metamaterials, which can perform a variety of functions such as sound absorbers and stealth devices [1]. As a typical acoustic metamaterial, the Pentamode metamaterial ("PM" material) appeared in recent years. Milton [2] independently proposed the "pentamode" theory in 1995 from the perspective of theoretical analysis. The main characteristic of this material is that only one of the six diagonal elements of the elastic coefficient matrix is non-zero, and the external appearance is that it can only propagate one type of elastic wave. PM material has special physical properties, and its bulk modulus is much greater than the shear modulus, leading to a significantly greater ratio of bulk modulus to shear modulus than natural materials. In addition, the PM structure is easily deformed because of the great ratio value, and it is difficult to be compressed, which means that they can exhibit fluid-like properties. The PM material is a new kind of mechanical metamaterial. Due to its specially mechanical properties, it can effectively control the propagation of acoustic waves and can be used to achieve the stealth of acoustic free space.

For a typical PM material, its most basically structural configuration is a diamond-like structure, which can be simplified as a honeycomb structure in two dimension case. At present, the research of this kind of diamond-like structure is more common. Kadic [3] successfully creates a solid model by the 3D printing technology. The paper utilizes the PM material configuration proposed by Xu Bing [4]. Although it also adopts a symmetrical double cone rod structure and constitutes a micro-truss PM material, the symmetry of the micro-truss material cell makes it easy to design and optimize material parameters of structures. The design of this PM material structure is represented in Fig.1, where Fig.1 (a) is the initial design structure of the PM material, the shaded area is an annular area composed of the isotropic PM material with its material properties similar to surrounding fluids. Fig.1 (b) is the resulted design structure, the shadow area is composed of anisotropic PM material. Fig.1 is also to describe the basic design process of the PM material structure, mainly by iteratively adjusting the micro truss size parameters to gradually change the wave propagation characteristics to meet the demand of the design.

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Figure 1. Acoustic structure design procedure

The content of this paper can be divided into two parts. In the first part, the transformation acoustics method is briefly introduced and the relationship between the effective elastic modulus of the PM material and the size parameters of the micro truss material unit is investigated and established. In the second part, the structural optimization model is constructed for the design of the PM material structure, and the final design structure is obtained from the initial PM material structure by optimizing or iteratively modifying. During the design process, the structure always keeps a PM material structure. At Final, a numerical example of structural design of PM material is given.

2. Acoustic Transformation

According to the transformation acoustic theory [6-7], the material property of the design structure relies on the coordinate transformation from an original domain Ω to a current domain ω , which is given by the point-wise deformation $X \in \Omega \rightarrow x \in \omega$. The outer boundaries of the two spaces are the same, that is $R = r = b$, and the inner boundaries of the two annular areas are c and a , the transformation between the original domain and current domain can be written as:

$$R = f(r) = c \frac{b-r}{b-a} + b \frac{r-a}{b-a} \quad (1)$$

Using the notation from the theory of finite elasticity, the deformation gradient is defined as $F = \nabla_x x$, and the Jacobian of the deformation gradient is $J = \det F$. The polar decomposition is $F = VR$, where R is proper orthogonal ($RR^T = R^T R = I, \det R = 1$) and the left stretch tensor V is the positive definite solution of $V^2 = B$, here B is the left Cauchy-Green tensor $B = FF^T$.

Given a pair of transformation domains, the design material is not unique. For instance, the inertial material can be chosen and defined by the density tensor $\rho = \rho_0 JB^{-1}$ and bulk modulus $K = K_0 J$. And another possible material chosen is the PM material if the coordinate transformation satisfies the condition that $\text{div} hV = 0$ for a scalar function $h(x)$, especially, in the case of $R = I$, the PM material property can be defined as the density $\rho = \rho_0 J^{-1}$ and elastic tensor

$C = KS \otimes S$ in terms of $K = K_0 J$ and $S = J^{-1}V$.

Furthermore, if the coordinate transformation is symmetric about original point, the PM material parameters in domain ω will possess the orthogonal bulk modulus and the isotropic density, which can be described in a polar coordinate system as follows,

$$K_r = C_{rrrr} = K_0 J^{-1} B_{rr}, K_{\perp} = K_0 J^{-1} B_{\theta\theta}, \rho = \rho_0 J^{-1} \quad (2)$$

Where subscript θ denotes the direction orthogonal to the radial direction.

3 Effect of Structural Parameters of cell on effective Elastic Modulus

Figure 2. Solid model of uniform unit cell

Fig.2 is the unit cell configuration and the partially enlarged view of the junction of rods of the PM material configuration. Here a uniform unit cell is illustrated with the rod length of l and the cell lattice constant of a , satisfying $l = (\sqrt{3}/2 + \sqrt{3})a$. Moreover, in order to satisfy the ratio of the effective elastic modulus (B/G) is sufficiently large, the symmetrical double cone structure is also employed instead of an unstable hinge connection, where the large diameter of the rod is D , the small diameter of the rod is d , and the angle between the bar of the circular table and the axis is 30° .

Firstly, the effective elastic modulus of the unit cell model is calculated by using the asymptotic homogenization method. Here, the material is chosen as rubber and its material properties are $E = 7.8 \text{ MPa}$, $\mu = 0.47$, $\rho = 10^{-3} \text{ g/mm}^3$; and the cell parameters are $a = 2 \text{ mm}$, $D = 0.1 \text{ mm}$, $d = 0.01 \text{ mm}$. And the elastic modulus can be obtained, as shown in Table 1.

Table 1. Effective elastic modulus of unit cell (10^4 Pa)

E_{1111}	E_{2222}	E_{3333}	E_{2323}	E_{3131}	E_{1212}
2.196	2.197	2.194	0.01186	0.01191	0.01185

E_{1111} , E_{2222} and E_{3333} represent the effective elastic moduli in the x, y and z directions of the unit cell, respectively, E_{2323} , E_{3131} and E_{1212} represent the shear moduli in the x, y and z directions of the unit cell. Table 1 shows that the ratio of the bulk modulus to the shear modulus of the material is large enough to satisfy the properties of the PM material.

3.1 Effect of Rod Radius of Unit Cell on Elastic Modulus E and Elastic Modulus Ratio r

Because only the tangential and radial elastic moduli (E_{yy} and E_{zz}) are considered during the optimization process, and their ratio is defined as $r = E_{zz} / E_{yy}$. The changes of structural parameters of unit cells are shown in Fig.3. Where, Fig.3 (a) shows the change of big radius of the rod parallel to the y direction, and Fig.3 (b) shows the change of small radius of the rod parallel to the y direction.

Figure 3. Variation of rod radius in unit cells

As shown in Fig.4, it can be seen that the effect of the big radius of the rod on E_{yy} and E_{zz} is basically the same. To begin with, E_{yy} and E_{zz} increase rapidly and then gradually slow down and tend to be stable. While increasing the small radius of the rod, E_{yy} and E_{zz} increase linearly. For the elastic modulus ratio r , the change according to the rod radius is not obvious, and it is similar to a horizontally straight line, which also provides us the convenience of adjusting the elastic modulus and the elastic modulus ratio independently.

(a) Big radius of the rod

(b) Small radius of the rod

Figure 4. Effect of the rod radius on E_{yy} , E_{zz} and r

3.2 Effect of Shape Ratio k of unit Cells on Elastic Modulus E and Elastic Modulus Ratio r

Figure 5. Unit cells with different shape ratios

Fig.5 shows unit cells with different shape ratios k , where the rod length along the z direction is changed in order to obtain different k . The shape ratio is defined as the ratio of the rod length along the z direction to the rod length along the y direction, that is $k = l_z / l_y$. Fig.5(a) shows the unit cell with $k = 0.336$, where the rod length is 0.31mm and it is the shortest; Fig.5(b) represents for the case of $k=0.7$; Fig.5(c) stands for the case of $k = 1$ and the unit cell is isotropic.

As shown in Fig.6, E_{yy} gradually decreases in accordance with the increase of k , in contrast, E_{zz} increases in accordance with the increase of k , but the difference between E_{yy} and E_{zz} decreases according to the increase of k . As a result, the elastic modulus ratio r increases with the increase of k , approximating a parabolic relationship.

Therefore, we can obtain a specific anisotropic unit cell structure, and obtain a specific elastic modulus ratio r by changing the shape ratio k .

Figure 6. Effect of shape ratios on E_{yy} , E_{zz} and r

3.3 Effect of Inner Side Length l of Unit Cells on Elastic Modulus E and Elastic Modulus Ratio r

Figure 7. Unit cell with different inner side length l

Fig.7 illustrates unit cells with different inner side length l . It can be seen that these cells are still isotropic. Therefore, their effective elastic modulus E_{yy} and E_{zz} are same, and the elastic modulus ratio r is approximately equal to 1. As shown in Fig.8, E_{yy} and E_{zz} are both increase as the inner side length l increases, and their relationship is approximate to a parabolic curve. In order to obtain an anisotropic PM material unit cell with a certain elastic modulus ratio r , it is best to adjust the shape ratio k of the unit cell. For different inner side length, the fitting functions between r and k are presented in Table 2.

Figure 8. Effect of the inner side length on E_{yy} , E_{zz} and r

Table 2. Fitting functions between r and k for different l

$l(\text{mm})$	$r(k)$
0.9282	$r(k) = 0.2397k^2 + 0.4608k + 0.2991$
1	$r(k) = 0.3907k^2 + 0.3707k + 0.2793$
1.2	$r(k) = 0.4011k^2 + 0.4455k + 0.1675$
1.4	$r(k) = 0.56k^2 + 0.3513k + 0.1089$
1.6	$r(k) = 0.4012k^2 + 0.516k + 0.01178$

4 Optimization Model for the Design of PM Material Structures

The design of PM material structures is described as an optimization process in which an initial isotropic PM material structure is gradually transformed into the final anisotropic PM material structure. This process is achieved by updating the structural dimensions of the PM material unit cell iteratively, and the updating procedure can be easily realized by adjusting the nodal coordinates of micro trusses, because the rod length, balance matrix and internal force of unit cells are all the function of the node coordinates.

The optimization model of the PM material structure, as shown in Fig.1, can be described as Eq.(3), here the stability of the PM material structure has been enhanced during the optimization process, and the balance matrix equation of micro truss is added to the constraints [4].

$$\begin{cases} \min & f(x_1, x_2, \dots, x_m) = \frac{1}{r^2} \\ \text{subject to} & -l_{ji} + \delta_i \leq 0 \quad i = 1, 2, \dots, n \\ & r_i = E_{\perp i} / E_{\theta i} = r(l_{ji}, l_{yi}) = r(k_i) \\ & -t \leq 0 \\ & At = Q \\ & x_j \in R^m \quad j = 1, 2, \dots, m \end{cases} \quad (3)$$

Where the objective function is designed to be the reciprocal of the square of the radius in order to obtain the maximum internal space; the value δ_i is a limit value of the rod length, i represents the number of layers, the elastic modulus can also be controlled by the cross-section radius of the rod, so we can finally get the specific unit cell size including the section radius and the rod length; A represents the balance matrix, t denotes the internal node force vector, Q stands for the external node force vector, and R^m represents the space which the nodes are belong to. And the entire optimization process can be expressed as follows:

Step 1: Set the initial micro truss node coordinates and the initial domain mapping relationship Eq. (1).

Step 2: Use the mapping relationship and the elastic modulus formulae by transformation acoustics Eq.(2) to calculate K_{θ} and $K_{\perp}$.

Step 3: Compute the elastic modulus ratio $r = K_{\perp} / K_{\theta}$ and find the fitting function $r = r(k)$ between the elastic modulus ratio and the shape ratio according to Table 2.

Step 4: Carry out the optimization (3), and update the domain mapping relationship.

Step 5: Repeating step 2 to step 5 until the shape ratio converges.

Step 6: The optimized results needs to be post-processed further so that the effective elastic modulus and the effective density all satisfy the acoustic transformation formulae.

5 Numerical Examples

An initial non-uniform annular structure composed of PM material is given as follows, there are 4 layers micro trusses in the structure, and each layer consists of 32 material unit cells with the same size. From the inside to the outside, the unit cell size of each layer is sequentially scaled up and the proportional coefficients are all 1.2185. The inner side length l is assigned as 1.6 mm , and the other structure parameters are set according to the dimensions of the basic unit cell with shape ratio $k=1$, shown in Fig.5. The design aim is to expand the inner radius of structure and compress the PM material outward along the radial direction, in order to obtain a large inner space. Since only the tangential and radial wave propagation are considered, the node coordinates, the rod length and the tilt angle of rods along the axial direction will remain unchanged during the optimization process.

The optimization results are shown in Fig.9, where Fig.9 (a) represents axonometric drawing of the initial structure; Fig.9 (b) is the top view of the initial structure; Fig.9 (c) represents axonometric drawing of the final design structure; Fig.9 (d) is the top view of the final design structure. Obviously, the internal space in the final design structure is expanded after the optimization. In addition, some parameters for the initial structure and the final design structure are given in Table 3, and the elastic modulus ratio r and density ρ of PM material unit cells along the structure radius are described in Fig.10.

Table 3. Parameters for the initial and final structures

layer number	Nodal number (N)		Number of rods	Balance matrix (A)	Initial Inner radius R : mm	Result Inner radius R : mm	Inner radius ratio
	Fixed node	Free node					
4	64	1280	2560	$A_{3584 \times 2560}$	10.1532	14.4225	1.4205

Figure 9. Optimization design of an annular structure

Figure 10. Elastic modulus ratio r and effective density ρ

In order to reduce the computing complexity, the elastic modulus ratio r is the only parameter related to the material property of the structure, which is considered during the optimization process. Meanwhile, the elastic modulus ratio is substituted by the shape ratio of material unit cells in the optimization model (3). As a result, the effective density ρ is maintained to be a constant during the optimization process, as shown in Fig.10. Therefore, in order to make the final result satisfy the acoustic transformation theory, the optimization result needs to be possessed further. For example, the Jacobian J and the bulk modulus K_0 in Eq.(2) are taken into account, and the structure sizes of material unit cells are designed and determined, including the large diameter of rods, the small diameter of rods except the shape ratio of unit cell, so that the effective elastic modulus and the effective density are consistent with Eq.(2).

6 Conclusion

An optimization method is presented for the design of a structure with PM materials. With incorporating the transformation acoustics algorithm into the optimization procedure, the optimization model converts material constraints into size constraints of micro-truss by employing the relationship between the effective elastic moduli of materials and the structural parameters of micro-truss unit cells. As a result, the design of PM material structures can be realized as a problem of structure size optimization.

The influence of the structural parameters of micro-truss unit cells on the effective elastic modulus and

density is investigated, and the adjustment of the shape ratio of unit cells is a simple way to construct anisotropic PM material.

In the optimization model, only the elastic modulus ratio is considered in the material properties, and then it is substituted by the shape ratio of material unit cells. By this way, the computing complexity is efficiently reduced. And the effective elastic modulus and density can be easily modified during optimization.

In addition, the stability of PM material structure can be ensured by constraining the balance matrix equation and rod forces in the optimization process [4].

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