A variational method in the problem of convective heat transfer near the vertical plane of the outer building fence in terms of free air movement

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Abstract. An unconventional method for solving the heat transfer problem for free convection near the vertical surfaces of the enclosing structures is proposed. It is based on the laws of thermodynamics of irreversible processes. A variational principle is used to find the unknown functions that analytically express the velocity fields and other flow quantities in the boundary layer near the vertical walls. A variational formulation of the hydrodynamic and thermal boundary layer near a vertical plane surface in conditions of free convection is given. It is given an analytical solution of the problem of the distribution of the vertical component of velocity and temperature in the wall boundary layer in the laminar flow. Theoretical formulas for the thickness of the boundary layer are obtained, as well as for the local and mean values of convective heat transfer coefficient.

1 Introduction

Most of the phenomena occurring in nature and technology are associated with nonequilibrium states and irreversible processes, and therefore they are described by classical thermodynamics only in the first approximation. The transition from thermodynamics (more correctly from thermostatic) to equilibrium states to the thermodynamics of irreversible processes marks a serious progress in the development of a number of sciences. This section of thermodynamics finds an increasing application in various fields of modern physics, chemistry and technology.

In this paper we propose a non-traditional modern method for solving the heat transfer problem for free convection near a vertical wall, based on the laws of thermodynamics of irreversible processes. A variational principle is used to find the unknown functions that analytically express the velocity and flow temperature fields in the boundary layers of the air flow near the vertical plane.

2 Methods

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Let us consider the process of nonisothermal flow of a viscous liquid near a vertical wall having a constant temperature t_w . The flow temperature far from the wall is the same and is equal to t_0 . We set $t_w > t_0$, where t_0 is the air temperature in the unperturbed flow.

As it nears the wall, the stream temperature rises, and its density decreases.

As a result of the motion of the heated flow layers near the wall, the hydrodynamic and thermal boundary layers $\delta(x)$ are formed. The velocity $W_x(y)$ changes from a zero value on the wall to a maximum value $W_m(y)$, and then again decreases to zero at the outer boundary $\delta(x)$. The air temperature decreases from $t = t_w$ to a value of t_0 in the unperturbed medium when $y > \delta(x)$. The profiles of the velocity and temperature fields are shown in the Fig. 1.

Fig. 1. Model of velocity and temperature distribution in the boundary layer.

The dependence of the flow density ρ on the temperature will be taken as linear

$$\rho = \rho_0(1 - \beta \vartheta) \quad (1)$$

where $\vartheta = t - t_0$, ρ_0 – air density at t_0 ; β – coefficient of volumetric expansion of air. Other physical parameters of air will be considered as constant.

In the boundary layer let us single out an elementary volume of liquid bounded by the planes AA_1 and BB_1 , the wall plane and parallel plane CD at an arbitrary distance from the wall surface. The width of the layer in the direction perpendicular to the plane of the drawing will be equal to one. Consequently, the value of the allocated volume is

$$dV = (\delta - y)dx \cdot 1 \quad (2)$$

At an arbitrary distance y from the wall surface, we draw a plane CD parallel to it. The frictional force F_{fr} acting in the indicated plane on the selected element is expressed by the formula

$$dF_{fr} = \tau dx \cdot 1 \quad (3)$$

where τ – the shear stress or the frictional stress in the indicated plane.

The second work of friction will be equal to the product of this force by the rate of liquid deformation in the plane under consideration:

$$dE_{fr} = \tau \frac{\partial w_x}{\partial y} dx dy \cdot 1 \quad (4)$$

Formula (4) expresses the amount of energy that is dissipated (passes into thermal energy and disperses). The flow of this energy is directed toward lower temperatures to the outer boundary of the wall layer, i.e. in the element $C1, D1, B1, A1$. It should be noted that, in addition to the indicated energy (4), there will be losses associated with changes in the

velocities $\frac{\partial w_x}{\partial x}$, $\frac{\partial w_y}{\partial y}$ and $\frac{\partial w_z}{\partial z}$. However, because of the smallness of these quantities compared with $\frac{\partial w_x}{\partial y}$ they can be neglected.

The second change in the energy density of A_1, B_1, DC due to dissipation in the plane CD will be

$$de_{fr} = \frac{\tau}{\delta - y} \frac{\partial w_x}{\partial y} dy. \quad (5)$$

Accordingly, the integral characteristic of viscous friction in the entire selected element $AA_1 BB_1$ will have the form

$$e_{fr} = \int_0^{\delta} \frac{\tau}{\delta - y} \frac{\partial w_x}{\partial y} dy. \quad (6)$$

For a laminar flow in a boundary layer, the quantity τ can be determined according to Newton's law:

$$\tau = \mu \frac{\partial w_x}{\partial y},$$

where μ - dynamic flow viscosity.

Consequently, the integral (6) can be rewritten in the following form:

$$e_{fr} = \int_0^{\delta} \frac{\mu}{\delta - y} \left(\frac{\partial w_x}{\partial y} \right)^2 dy. \quad (7)$$

From the theory of the boundary layer it is known that the distribution of velocity w_x along the coordinate y is given by

$$w_x = w_m(x) \cdot \varphi(\eta), \quad (8)$$

where $w_m(x)$ - the maximum velocity in the layer; $\varphi(\eta)$ - the required function; $\eta = \frac{y}{\delta}$.

After substituting (8) into the integral (7)

$$e_{fr} = \frac{\mu w_m^2}{\delta^2} \int_0^1 \frac{(\varphi')^2}{1 - \eta} d\eta, \quad (9)$$

where $\varphi' = \frac{d\varphi}{d\eta}$.

The second change in the density of the potential energy of the selected element of the boundary layer will be determined by the work of the lifting forces:

$$e_p = \frac{1}{\delta} g \int_0^{\delta} (\rho - \rho_0) w_x dy = -g \rho_0 w_m \beta \vartheta_m \int_0^1 \varphi \psi d\eta, \quad (10)$$

where $\psi = \frac{y}{\vartheta_m}$; $\vartheta_m = t_{cm} - t_0$.

From the thermodynamic point of view, in the flow of a viscous liquid, degradation occurs - energy depreciation, as in all other real processes taking place in nature. However, nature is not "wasteful". Internal nonequilibrium processes always act in a direction that causes a decrease in the rate of entropy increase. This determines the stability of stationary states.

In the thermodynamics of irreversible processes, this law is formulated as follows: as the system moves to a stationary-non-equilibrium state, the value of the incremental increase in entropy decreases and, when the stationary-nonequilibrium state is reached, it assumes the smallest value compatible with external constraints [1-3]. Thus, the fields of physical values for stationary nonequilibrium states can be investigated on the basis of the extremal principle. If we neglect the change in the kinetic energy of the flow, then following the well-known in mechanics Helmholtz principle, we can assert that the integral in the steady-state process will have a minimum.

$$\int_0^1 (e_p + e_{fr}) d\eta. \quad (11)$$

Taking into account expressions (9) and (10), it will be

$$\int_0^1 \left(-a_1 \varphi \psi + a_2 \frac{\varphi'^2}{1-\eta} \right) d\eta = \min, \quad (12)$$

where $a_1 = g \rho_0 w_m \beta \vartheta_m$, $a_2 = \frac{\mu w_m^2}{\delta^2}$.

Thus, the problem of distribution of velocity $w_x(y)$ in the boundary layer can be reduced to the problem of the variations calculus [4-8], namely, to the study of the functional (12) for an extremum under the following boundary conditions

$$\begin{aligned} 1) & \quad w_x(0) = 0; \\ 2) & \quad w_x(\delta) = 0; \\ 3) & \quad \left. \frac{\partial w_x}{\partial y} \right|_{y=\delta} = 0; \\ 4) & \quad \left. \frac{\partial^2 w_x}{\partial y^2} \right|_{y=\delta} = \frac{g(\rho_{cm} - \rho_0)}{\mu}. \end{aligned}$$

With respect to the required function $\varphi(\eta)$ these conditions can be rewritten in the following form:

$$1) \quad \varphi(0) = 0;$$

$$\begin{aligned} 2) \quad & \lim \varphi = 0 \text{ when } \eta \longrightarrow 1; \\ 3) \quad & \left. \frac{d\varphi}{d\eta} \right|_{\eta=1} = 0; \\ 4) \quad & \left. \frac{d^2\varphi}{d\eta^2} \right|_{\eta=1} = \frac{\delta^2 g(\rho_{cm} - \rho_0)}{\mu w_m}. \end{aligned}$$

The fourth boundary condition follows from the equation of a viscous fluid motion in the boundary layer:

$$\rho w_x \frac{\partial w_x}{\partial x} + \rho w_y \frac{\partial w_x}{\partial y} = \mu \frac{\partial^2 w_x}{\partial y^2} + g(\rho_{cm} - \rho_0). \quad (13)$$

The necessary condition for the extremum of the integral (12) (the Euler-Lagrange equation) for the case under consideration has the following form:

$$\frac{\partial F}{\partial \varphi} - \frac{d}{d\eta} \left(\frac{\partial F}{\partial \varphi'} \right) = 0, \quad (14)$$

where $F = -a_1 \varphi \psi + a_2 \frac{\varphi'^2}{1-\eta}$ - subintegral function.

Performing the operations of differentiation corresponding to the expression (14), we obtain the following differential equation:

$$\varphi'' + \frac{\varphi'}{1-\eta} = -a_3(1-\eta)\psi, \quad (15)$$

$$\text{where } a_3 = \frac{a_1}{2a_2} = \frac{\rho_0 g \beta v_m \delta^2}{2\mu w_m}. \quad (16)$$

In the first approximation $\psi = (1-\eta)^2$, consequently, equation (15) takes the following form:

$$\frac{\varphi'' + \varphi'}{(1-\eta)} = -a_3(1-\eta)^3. \quad (17)$$

Denote $\varphi' = p(\eta)\varphi$, then we obtain

$$\frac{p' + p}{(1-\eta)} = -a_3(1-\eta)^3. \quad (18)$$

Equation (18) has the following general solution:

$$p = (1-\eta) \left(c_1 - a_3 \eta + a_3 \eta^2 - \frac{a_3}{3} \eta^3 \right). \quad (19)$$

Since $p = \varphi'$, integrating this equation once again with respect to the variable η , we obtain

$$\varphi = c_1 \eta \left(1 - \frac{1}{2} \eta \right) - a_3 \eta^2 \left(\frac{1}{2} - \frac{2}{3} \eta + \frac{1}{3} \eta^2 - \frac{1}{15} \eta^3 \right). \quad (20)$$

From the second boundary condition $\varphi(1) = 0$ we obtain $c_1 = \frac{a_3}{5}$.

Consequently:

$$\varphi = a_3 \eta \left(\frac{1}{5} - \frac{3}{5} \eta + \frac{2}{3} \eta^2 - \frac{1}{3} \eta^3 + \frac{1}{15} \eta^4 \right). \quad (21)$$

Let us now clarify the temperature distribution in the boundary layer. To do this, we consider the differential equation of the layer energy:

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} = a \frac{\partial^2 t}{\partial y^2}. \quad (22)$$

Taking into account the continuity equation it will be

$$w_y = \int_0^y \frac{\partial w_y}{\partial y} dy = - \int_0^y \frac{\partial w_x}{\partial x} dy. \quad (23)$$

Since

$$\begin{aligned} \frac{\partial w_x}{\partial x} &= w_m \frac{\partial \varphi}{\partial x} + \frac{dw_m}{dx} \varphi = - \frac{w_m}{\delta} \frac{d\delta}{dx} \eta \varphi' + \varphi \frac{dw_m}{dx}, \\ \text{then } w_y &= w_m \frac{d\delta}{dx} \int_0^\eta \varphi' \eta d\eta - \frac{dw_m}{dx} \delta \int_0^\eta \varphi d\eta = w_m \frac{d\delta}{dx} (\varphi \eta - f) - \frac{dw_m}{dx} \delta f, \end{aligned} \quad (24)$$

where $f = \int_0^\eta \varphi d\eta$.

We write the obvious relations:

$$\begin{aligned} \frac{\partial t}{\partial x} &= - \frac{w_m}{\delta} \frac{d\delta}{dx} \eta \psi; \\ \frac{\partial t}{\partial y} &= \frac{w_m}{\delta} \psi'; \\ \frac{\partial^2 t}{\partial y^2} &= \frac{w_m}{\delta^2} \psi''; \end{aligned} \quad (25)$$

where $\psi' = \frac{d\psi}{d\eta}$; $\psi'' = \frac{d^2\psi}{d\eta^2}$.

Taking into account expressions (24), (25), the energy equation (22) can be represented in the following form:

$$\psi'' + \frac{1}{a} \left(w_m \delta \frac{d\delta}{dx} + \delta^2 \frac{dw_m}{dx} \right) f \psi' = 0. \quad (26)$$

We denote:

$$\frac{1}{a} \left(w_m \delta \frac{d\delta}{dx} + \delta^2 \frac{dw_m}{dx} \right) = A. \quad (27)$$

$$\text{Then } \psi'' + A f \psi' = 0; \quad (28)$$

$$f = \int_0^\eta \varphi d\eta = a_3 \left(\frac{1}{10} \eta^2 - \frac{1}{5} \eta^3 + \frac{1}{6} \eta^4 - \frac{1}{15} \eta^5 + \frac{1}{90} \eta^6 \right). \quad (29)$$

We seek the solution of (28) from

$$\psi = b_0 + b_1 \eta + b_2 \eta^2 + b_3 \eta^3 + \dots \quad (30)$$

We find the derivatives ψ' and ψ'' and substitute them in equation (28). Then we receive:

$$(2b_2 + 6b_3\eta + 12b_4\eta^2 + \dots) + A a_3 \left(\frac{1}{10} \eta^2 - \frac{1}{5} \eta^3 + \frac{1}{6} \eta^4 - \frac{1}{15} \eta^5 + \frac{1}{90} \eta^6 \right) (b_1 + 2b_2\eta + 3b_3\eta^2 + 4b_4\eta^3 + \dots) = 0. \quad (31)$$

After multiplication of series, it will be:

$$(2b_2 + 6b_3\eta + 12b_4\eta^2 + 20b_5\eta^3 + 30b_6\eta^4 + 42b_7\eta^5 + \dots) + Aa_3 \left(\frac{1}{10}b_1\eta^2 - \frac{1}{5}b_1\eta^3 + \frac{1}{6}b_1\eta^4 - \frac{1}{15}b_1\eta^5 + \dots \right) + Aa_3 \left(\frac{1}{5}b_2\eta^3 - \frac{2}{3}b_2\eta^4 + \frac{1}{3}b_2\eta^5 - \dots \right) + Aa_3 \left(\frac{9}{10}b_3\eta^4 - \frac{3}{5}b_3\eta^5 + \dots \right) + Aa_3 \left(\frac{2}{5}b_4\eta^5 - \dots \right) = 0. \quad (32)$$

Comparing the coefficients for the same powers of η , we obtain the following equalities:

$$2b_2 = 0;$$

$$6b_3 = 0;$$

$$\frac{1}{10}Aa_3b_1 + 12b_4 = 0;$$

$$-\frac{1}{5}Aa_3b_1 + 20b_5 = 0;$$

$$\frac{1}{6}Aa_3b_1 + 30b_6 = 0.$$

It follows:

$$b_2 = 0;$$

$$b_3 = 0;$$

$$b_4 = -\frac{1}{120}Aa_3b_1;$$

$$b_5 = \frac{1}{100}Aa_3b_1;$$

$$b_6 = -\frac{1}{180}Aa_3b_1.$$

As $\psi(0) = 1$, then $b_0 = 1$.

Therefore:

$$\psi = 1 + b_1\left(\eta - \frac{Aa_3}{120}\eta^4 + \frac{Aa_3}{100}\eta^5 - \frac{Aa_3}{180}\eta^6 + \dots\right). \quad (33)$$

Restricting to the first five terms of the series, from the boundary condition $\psi'(0) = 0$:

$$Aa_3 = 60.$$

From another boundary condition $\psi'(1) = 0$:

$$b_1 = -\frac{30}{23}.$$

Thus, we have the following refined expression for the temperature distribution in the boundary layer:

$$\psi = 1 - \frac{1}{23}(30\eta - 15\eta^4 + 18\eta^5 - 10\eta^6). \quad (34)$$

As it nears the wall, the accuracy of the solution increases. This is especially important, since the heat transfer under consideration is mainly determined by the value of the flow temperature gradient at the wall surface. Substitution of this expression into the differential equation (15) leads to a solution that differs little from the solution (21) for φ found earlier, so we will not further refine the function ψ (2). Let us find the ordinate value η_m corresponding to the maximum velocity w_m in the boundary layer.

According to the formula (21)

$$\psi'(\eta_m) = a_3 \left(\frac{1}{5} - \frac{6}{5}\eta_m + 2\eta_m^2 - \frac{4}{3}\eta_m^3 + \frac{1}{3}\eta_m^4 \right). \quad (35)$$

Then $\eta_m = 0,26$.

As $\varphi(\eta_m) = 1$, then

$$0,26a_3 \left(\frac{1}{5} - \frac{3}{5}\eta_m^2 + \frac{2}{3}\eta_m^3 - \frac{1}{3}\eta_m^4 + \frac{1}{15}\eta_m^5 \right) = 1.$$

Therefore:

$$a_3 = 46,3;$$

$$A = \frac{1}{a} \left(w_m \delta \frac{d\delta}{dx} + \delta^2 \frac{dw_m}{dx} \right) = 1,3.$$

Taking into account the notations (22) and (27) adopted above, we obtain the following system of equations:

$$\begin{cases} a_3 = \frac{\rho_0 g \beta \vartheta_m \delta^2}{2\mu w_m} = 46,3 \\ A = \frac{1}{a} \left(w_m \delta \frac{d\delta}{dx} + \delta^2 \frac{dw_m}{dx} \right) = 1,3 \end{cases} \quad (36)$$

From the first equation of this system follows:

$$w_m = \frac{\rho_0 g \beta \vartheta_m \delta^2}{92,6\mu}; \quad (37)$$

$$\frac{dw_m}{dx} = \frac{\rho_0 g \beta \vartheta_m \delta}{46,3\mu} \cdot \frac{\delta}{dx}. \quad (38)$$

Therefore, the second equation of the system (36) can be rewritten in the following form:

$$B \delta^3 \frac{d\delta}{dx} = 1,3a, \quad (39)$$

where $B = \frac{3\rho_0 g \beta \vartheta_m}{92,6\mu}$.

3 Results

Solving the differential equation (39), we obtain

$$\delta = 3,56 \left(\frac{g \beta \vartheta_m}{\nu a x} \right)^{-\frac{1}{4}} \quad (40)$$

where $\nu = \frac{\mu}{\rho}$.

Given that the Grashof number $Gr_x = \frac{g \beta \vartheta_m x^3}{\nu^2}$, and the Prandtl number $Pr = \frac{\nu}{a}$, we obtain

$$\frac{\delta}{x} = 3,56 (Gr_x Pr)^{-\frac{1}{4}}. \quad (41)$$

For the velocity w_m , we obtain the following expression:

$$w_m = \frac{\rho_0 g \beta \vartheta_m \delta^2}{92,6\mu}; \text{ or } w_m = 0,138 \left(\frac{g \beta \vartheta_m x}{Pr} \right). \quad (42)$$

The density of the heat flow at the surface of the wall is determined by the relation

$$q = -\lambda \left(\frac{\partial \vartheta}{\partial y} \right)_{y=+0} = -\frac{\lambda}{\delta} \vartheta_m \psi'_{\eta=+0}. \quad (43)$$

$$q = \alpha \vartheta_m. \quad (44)$$

Therefore, taking into account formula (41), we obtain the following expression for the local value of the heat transfer coefficient for a plane vertical surface:

$$\alpha = 0,365 \frac{\lambda}{x} (Gr_x Pr)^{\frac{1}{4}} \quad (45)$$

and criterial dependence:

$$Nu_x = 0,365 (Gr_x Pr)^{\frac{1}{4}}. \quad (46)$$

Performing the integration in the range from 0 to x , we find the expression for the average value of the heat transfer coefficient:

$$\bar{\alpha} = \frac{4}{3} \alpha,$$

that is the Nusselt number

$$Nu = 0,487 (Gr_x Pr)^{\frac{1}{4}}. \quad (47)$$

4 Conclusion

Obtained formulas are in better agreement with experimental data than the approximate solutions [9-15]. The applied method of solution, in contrast to traditional methods, more deeply determines the natural physical processes of hydrodynamics of liquid and gas, and can therefore be successfully applied to analysis and calculation of hydrodynamic and thermal characteristics of simple and complex structural elements in construction and power engineering.

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