Stability of digital feedback control systems

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Abstract. Specific problems arising, when Von Neumann type computer is used as feedback element, are considered. It is shown, that due to specifics of operation this element introduce pure lag into control loop, and lag time depends on complexity of algorithm of control. Method of evaluation of runtime between reading data from sensors of object under control and write out data to actuator based on the theory of semi-Markov process is proposed. Formulae for time characteristics estimation are obtained.
Lag time characteristics are used for investigation of stability of linear systems. Digital PID controller is divided onto linear part, which is realized with a soft and pure lag unit, which is realized with both hardware and software. With use notions amplitude and phase margins, condition for stability of system functioning are obtained. Theoretical results are confirm with computer experiment carried out on the third-order system.

1 Introduction

Including Von-Neumann type computer to digital control loops born many problems, main of which is the problem of gap from emergence a situation, which requires an adequate control system response, till the real action, affected onto the controllable object [1-3]. Time lag depends on a number of factors, such as computer architecture, clock frequency, instructions structure, operating environment, scheduling discipline, transactions order, mathematical foundation of control algorithms, etc.

Control algorithms have next specific features, which had been studied by several authors [4-6]:

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algorithms are a cyclic ones, i.e. they have start operator, but does not have the end operator;

quest of peripherals is realized by means of inclusion into algorithm special transaction management operators;

for an external observer selection of branch in places of algorithm ramification is a stochastic one, and probabilities of branching depend on a distribution of data processed;

for an external observer algorithm operators' run-time is, a random one, distribution function of time of operator execution depends on distribution of data processed.

Thus, a process of a deterministic algorithm interpretation with Von-Neumann type controller for external observer by several authors [5, 7, 8] is regarded as semi-Markov process with continuous time. Operators of an algorithm are considered as states of semi-Markov process. Interpretation of algorithm may be considered as sequence of state switches or wandering through the states of semi-Markov process.

Among the algorithm operators there are operators, which request sensors and operators, which request actuator of the object under control. Time intervals between semi-Markov process states, which are abstract analogues of operators mentioned and influence of intervals onto the quality of control in linear control systems is subject of following investigations.

2 General semi-Markov model of the control algorithm

The flowchart of digital feedback control system is shown on the Figure 1 [1, 2, 3, 5].

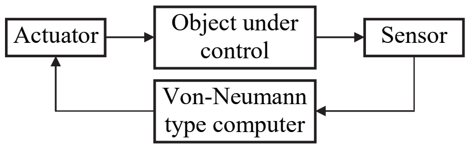

Fig. 1. Digital feedback control system.

The system includes object under control and the Von-Neumann type computer as a controller. The physical state of object under control is estimated by sensor, which generates data for computer processing. Output of Von-Neumann type computer through actuator affects on the object, so the feedback is closed.

Model of a system soft is the semi-Markov process [8-11]

$$\mu = \{A, \mathbf{h}(t)\}, \tag{1}$$

where $A = \{a_1, ..., a_j, ..., a_n, ..., a_J\}$ is the set of J states; $\mathbf{h}(t) = [h_{j,n}(t)]$ is the semi-Markov matrix of size $J \times J$; $h_{j,n}(t) = p_{j,n} \cdot f_{j,n}(t)$ is the weighed time density of residence in the state a_j before its switch into the state a_n; $p_{j,n}$ is the probability of switch from a_j into a_n;

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$f_{j,n}(t)$ - is the pure time density of residence in the state a_j with further switch into the state a_n.

Common structure of semi-Markov process, which simulates the soft of computer, is shown on the Figure 2 a. The process is ergodic, and is performed by the full graph with loops. From this structure all possible structures with *J* states may be obtained.

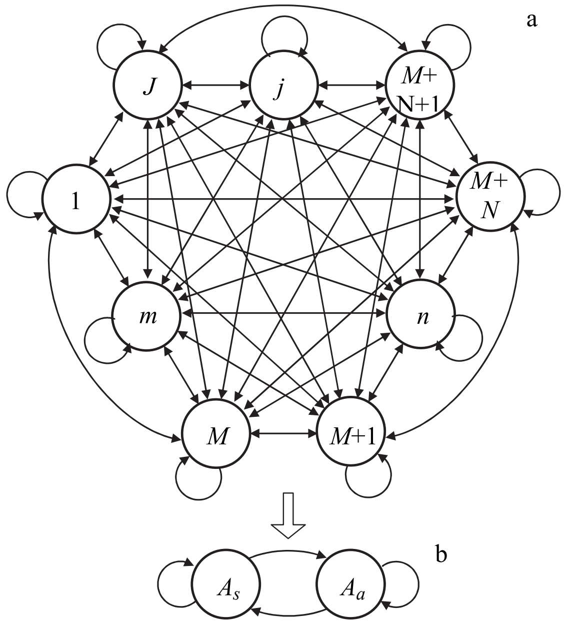

Fig. 2. Structures of Semi-Markov process; a - initial, b - after simplification.

In context of the task under solution the set of states *A* may be divided onto three disjoint subsets (classes):

$$A = A_s \cup A_a \cup A_o \quad A_s \cap A_a = \varnothing;$$

$$A_a \cap A_o = \varnothing; \; A_s \cup A_o = \varnothing, \tag{2}$$

where $A_s = \{a_1, ..., a_m, ..., a_M\}$ are states, which simulate sensor quests; $A_a = \{a_{M+1}, ..., a_n, ..., a_{M+N}\}$ are states, which simulate actuator quests; $A_o = \{a_{M+N+1}, ..., a_j, ..., a_J\}$ are states, which simulate other operators of cyclic algorithm.

As it is follows from the structure of semi-Markov process, quests to Von-Neumann computer peripherals, shown on the Figure 1, may be generated in the next cases:

when there are direct switches from states of subsets A_s or A_a to states of subsets A_s or A_a;

when there are wanderings from states of subsets A_s or A_a to states of subsets A_s or A_a through states of subset A_o.

To avoid second case one should to simplify the semi-Markov process (1) with use method, described in [12], as follows:

$$\mu \to \mu' \tag{3}$$

$$\mu' = \{A', \mathbf{h}'(t)\}, \tag{4}$$

$$A = A_s \cup A_a =$$
$$= \{a_1, ..., a_m, ..., a_M, a_{M+1}, ..., a_n, ..., a_{M+N}\}; \tag{5}$$

where $\mathbf{h}'(t) = [h'_{j,n}(t)]$ is the semi-Markov matrix of size $M+N \times M+N$; $h'_{j,n}(t) = p'_{j,n} \cdot f'_{j,n}(t)$.

In turn, simplified process (3) may be transformed into the process μ'', shown on the Figure 2 b as follows:

$$\mu' \to \mu''; \tag{6}$$

$$\mu'' = \left\{ \{A_s, A_a\}, \begin{bmatrix} h_{ss}(t) & h_{sa}(t) \\ h_{az}(t) & h_{aa}(t) \end{bmatrix} \right\} \tag{7}$$

where $h_{ss}(t) = p_{ss} f_{ss}(t)$ is the weighted time density of residence the process μ'' in one of subset A_s states with further switch into a state of the subset A_s; $h_{sa}(t) = p_{sa} f_{sa}(t)$ is weighted time density of residence the process μ'' in one of subset A_s states with further switch into a state of the subset A_a; $h_{as}(t) = p_{as} f_{as}(t)$ is weighted time density of residence the process μ'' in one of subset A_a states with further switch into a state of the subset A_s; $h_{aa}(t) = p_{aa} f_{aa}(t)$ is weighted time density of residence the process μ'' in one of subset A_a states with further switch into a state of the subset A_a.

Time density $f_{sa}(t)$ gives soughed-for lag in digital feedback control system. To define $f_{sa}(t)$ one should to evaluate probabilities of residence of semi-Markov process (4) in states of set (5) as follows [5, 7, 13]:

$$\pi_m = \frac{T_m}{\tau_m}, 1 \le m \le M, \tag{8}$$

where T_m is the time of a residence the process (4) in the state a_m; τ_m is the time of a return the process (4) at the state a_m;

$$T_m = \int_0^\infty t \sum_{j=1}^{M+N} h_{mj}(t) dt; \tag{9}$$

For estimation of τ_m one should split the state a_m onto ${}^b a_m$ and ${}^e a_m$. This may be done by means of transposition of *m*-th column of matrix $\boldsymbol{h}'(t)$ to ($M+N+1$)-th column. Emptied *m*-th column and ($M+N+1$)-th row should be fulfilled with zeros. As the result, matrix $\widetilde{\mathbf{h}}'(t)$, of size $(M+N+1) \times (M+N+1)$ should be formed, The expectation of the time of return the process into state a_m is as follows:

$$\tau_m = \int_0^\infty t \cdot L^{-1} \left({}^r\mathbf{I}_m \cdot \sum_{k=1}^\infty \{ L[\widetilde{\mathbf{h}}'(t)] \}^k \cdot {}^c\mathbf{I}_{M+N+1} \right) dt, \tag{10}$$

where ${}^{r}\mathbf{I}_m$ is the row vector of size $M+N+1$, m-th element of which is equal to one and other elements are equal to zeros; ${}^{c}\mathbf{I}_{M+N+1}$ is the column vector of size $M+N+1$, ($M+N+1$)-th element of which is equal to one, and all other elements are equal to zeros; $L[...]$, $L^{-1}[...]$ are direct an inverse Laplace transform correspondingly.

The time density $f_{sa}(t)$ among $\tau_m, 1 \le m \le M$ also depends on weighed time densities of switch from states of subset A_s into stets of subset A_a:

$$f_{sa}(t) = \frac{\sum_{m=1}^{M} \pi_m \sum_{n=M+1}^{M+N} h'_{m,n}(t)}{\sum_{m=1}^{1} \pi_m \sum_{n=M+1}^{M+N} p'_{m,n}}. \quad (11)$$

Next characteristics of lag are of interest when investigation of digital control systems stability: expectation T_{as}, dispersion D_{as}, minimum $T_{as\min}$ and maximum $T_{as\max}$ values [14]. They are as follows:

$$T_{as} = \int_0^\infty t f_{as}(t)dt; \; D_{as} = \int_0^\infty (t - T_{as})^2 f_{as}(t)dt;$$

$$T_{as\min} = \min_{m,n}\left[\arg h_{m,n}(t)\right];$$

$$T_{as\max} = \max_{m,n}\left[\arg h_{m,n}(t)\right]. \quad (12)$$

3 Stability of linear systems

Let us consider an influence of lag time $f_{sa}(t)$ on the stability of digital feedback control system, shown on the Figure 1 for the case, when model of object under control is the linear one. Control action is calculated as linear combination of sensor data, derivatives of the first and higher orders and integrals of the first and higher orders of sensor data (PID controller). Structural scheme of the linear system is shown on the Figure 3 [1, 2].

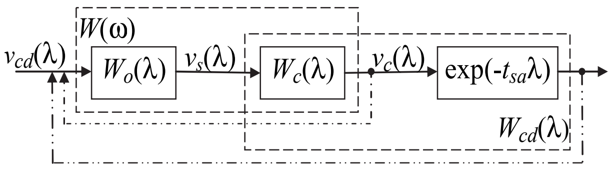

Fig. 3. Structural scheme of linear system.

On the Figure 3: $v_{cd}(\lambda)$ is the quasi-analogue computer output signal Laplace image; $v_s(\lambda)$ is the sensor signal Laplace image; $v_c(\lambda)$ is the quasi-analogue computer output signal Laplace image without lag; $W_o(\lambda)$ is the controllable object transfer function, including transfer functions actuator, object itself and sensor; $W_{\tilde{n}}(\lambda)$ is PID controller transfer function without lag [4, 15]; $\exp(-t_{sa}\lambda)$ is the transfer function of pure lag; $W(\lambda)$ – transfer function of system linear part; $W_{cd}(\lambda)$ is the transfer function of computer λ is the Laplace variable (the differentiation operator); t_{sa} is a random parameter, in accordance with section 2.

Substitution into transfer function $W_o(\lambda)$ $\lambda = i\omega$, where ω is the circular frequency; $i = \sqrt{-1}$ gives next form of the transfer function from the input of object under control till the output of computer without taking into account pure lag:

$$W(i\omega) = \frac{v_c(i\omega)}{v_{cd}(i\omega)} = \alpha(i\omega)\cdot\exp[i\varphi(\omega)], \quad (13)$$

where $\alpha(\omega) = |W(\omega)|$ is the amplitude-frequency response; $\varphi(\omega) = \arccos\frac{\operatorname{Im}[W(i\omega)]}{A(i\omega)}$ is the phase-frequency response.

Both amplitude-frequency response and phase-frequency response are shown on the Figure 4.

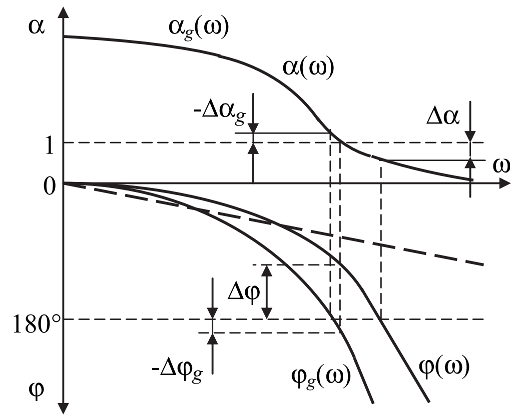

Fig. 4. Amplitude and phase frequency responses.

Differences $\Delta\alpha = 1 - \alpha(\omega)$ at the point $\varphi(\omega) = 180°$ and $\Delta\varphi = 180 - \varphi(\omega)$ at the point $\alpha(\omega) = 1$ give for a system without lag amplitude and phase margins respectively [1, 2].

Let us insert the unit $\exp(-t_{sa}\lambda)$ into the control loop. The amplitude response of the unit is equal to 1 in all frequency range [16, 17, 18, 19]. Phase response has the form of an inclined straight line

$$\varphi = t_{sa}\omega, \quad (14)$$

with the slope factor equal to t_{sa} (shown on the Figure 4 with dashed line).

In the system with lag values of straight line (14) are summed with values of phase response $\varphi(\omega)$. So amplitude $\alpha_g(\omega)$ and phase $\varphi_g(\omega)$ response of the system with a lag are as follows;

$$\alpha_g(\omega) = \alpha(\omega); \; \varphi_g(\omega) = \varphi(\omega) + t_{sa}\omega. \quad (15)$$

Summing the values of straight line with values of the curve increases a steepness of the curve. Increase of a steepness leads to decrease of amplitude $\Delta\alpha_g$ and phase $\Delta\varphi_g$ margins. If margins became negative the system losses stability.

So, to provide stability of the system one should to operate with use next simple method.

Method of a digital feedback control system stability estimation.

1) For the system $W(\lambda)$ define amplitude and phase margins $\Delta\alpha_g$ and $\Delta\varphi_g$.

2) For the control algorithm define $T_{as\max}$.

3) Check the next criteria for the system with a lag:

$$\varphi\{\arg[\alpha(\omega)=1]\}-T_{as\max}\cdot\arg[\alpha(\omega)=1]\leq 180^\circ\text{; } \quad (16)$$

$$\alpha\{\arg[\varphi(\omega)+T_{as\max}\omega=180^\circ]\}\leq 1. \quad (17)$$

If both conditions are satisfied, the system is stable, if not, there is necessary to change system hardware or software.

4 Example

As an example one would consider the system, in which linear part is performed by the transfer function

$$W(\lambda)=\frac{v_c(\lambda)}{v_{cd}(\lambda)}=\frac{0{,}2}{0{,}01\lambda^3+0{,}1\lambda^2+0{,}2\lambda+0{,}01}. \quad (17)$$

System response on the Heavisaid function is shown on the Figure 5.

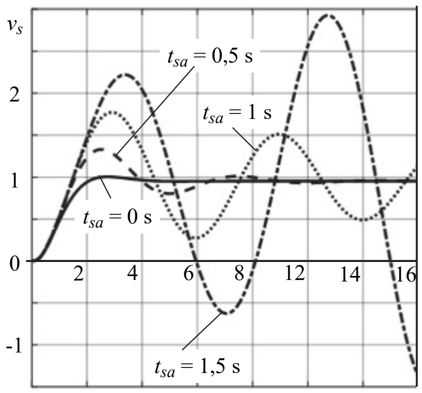

Fig. 5. Response of the system on the Heaviside function.

Response the system with closed loop without lag ($t_{sa}=0$) is shown with solid line. Response when $t_{sa}=0{,}5$ s is shown with dashed line< response when $t_{sa}=1$ s is shown with dotted line, and response when $t_{sa}=1{,}5$ s is shown with dash-dotted line. Criteria (15), (16) are met when $t_{sa}=0$, $t_{sa}=0{,}5$, $t_{sa}=1$ and are not met when $t_{sa}=1{,}5$. This is corfirmed by the graphs of transition processes. Process diverges when $t_{sa}=1{,}5$, and converges in all other cases.

5 Conclusion

Von-Neumann type computer lag time is the important factor, which should be taken into account when working out a digital control system, both hardware and software. Working out approach to evaluation a behavior of the system with pre-determined configuration opens new page in the theory and practice of system design. With use results expounded above one can to establish demands to software, especially to its time characteristics. This, in turn permits to limit computational complexity of control algorithm and facilitate decision making by software developers.

Further investigation in this area should be directed to working out practical recommendations for working out algorithm satisfied obtained limitations, and to finding links with other methods of software design.

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