The behavior of bilinear impact oscillators subjected to random forcings

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Abstract.
A multistable regime of a bilinear vibro-impact oscillator in the presence of noise has been studied, for which relatively lesser attractors coexist with a well-defined basin boundary structure in the phase space. The effect of adding parametric noise in this model has been investigated and the switching that the system undergoes between the basins of its different attractors has been examined.

1 Introduction

Strongly non-linear systems could exhibit multiple stable solutions for the same set of parameters. A consequence of this is that the stable solution to which the system converges at $t \to \infty$ is dictated by the basins of attraction on which the initial conditions of the system lie. The mathematical modelling of a physical system involves simplifying assumptions and approximations that introduce uncertainties into the model. These uncertainties are responsible for the deviations in the behaviour of the physical system with the predictions obtained from the mathematical model. This, in turn, can lead to erroneous predictions about the long term behaviour of a dynamical system. As these uncertainties are unknowns which can be characterized only in the probabilistic sense, it is important to carry out an analysis that enables predicting the long term behaviour in a probabilistic sense. This will enable identifying the most likely long term dynamical state of the system. For such multistable systems, a better understanding of their dynamics would help to demarcate the optimum operational conditions. As an example, for the classical tuned mass absorber, a device that is widely used to absorb energy and mitigate unwanted vibrations, optimal damping ability is achieved in the neighbourhood of its multistability zone [1]. However, among all the closely spaced co-existing solutions only one mitigates oscillations effectively. A slight disturbance would lead to other solutions that might amplify the amplitude of the base system, making its use counterproductive. Similarly, in systems with impacts, one of the stable solutions can ensure the desirable operation of the system while others may lead to vibration amplification and damage. Multistability in impacting systems has been studied in [2–4]. Vibro-impacting scenarios appear in a wide range of practical problems such as percussive drilling tools [5], print hammers [6] and vibro-impact moling systems [7]; most of these studies report multistability in deterministic scenarios. The present study aims at analysing the co-existing stable solutions of the bilinear impacting systems in the presence of slight disturbances.

On adding parametric noise with a high signal to noise ratio over the actual input forcing signal, the phenomenon of basin hopping as described in [2] has been observed. However unlike the studies discussed in the literature where the system gets stabilized indefinitely to the attractors that were previously present, the responses were seen to be flipping between the pre-existing attractors. Typically, the system showed such transition in behaviour for parameter regimes in the neighbourhood of its stability boundaries. Well defined basin boundaries cannot be defined in such scenarios. This is because, the presence of noise leads to alteration in the regimes of different attractors[8]. Thus, a detailed analysis of such scenarios is necessitated.

This paper is organized as follows. In Section 2 the model and the equations of motion for the impacting system have been presented. Section 3 investigates the coexistence of different regimes and their basins of attraction has been investigated. Section 4 summarizes the salient features arising from this study.

2 Problem Statement

A soft impacting system with bilinear stiffness has been considered; see Fig. 1 for a schematic diagram. It comprises of a mass-spring-damper system with mass M, spring stiffness k_1, and damping factor R_1. The cushioned impacting surface is modelled by a spring with stiffness k_2 and has been assumed to have negligible mass as compared to M. The equilibrium position of the left end of the spring k_1 is at zero and k_2 is at d. The mass M is subjected to a harmonic excitation of the form $u = A\sin(\Omega t)$ where Ω is the angular frequency. The state variable y is the elongation of the spring from the unstretched position. Initially when both the springs are relaxed, the mass is at a

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distance of d from the impacting surface. Let $\mathbf{Y} = [y; \dot{y}]^T$ be the state vector with $\dot{y}$ representing the velocity (in m/s) of the mass. The dynamics of the mass M are given by the Eq. 1.

$$M\ddot{y} = \begin{cases} -R_1\dot{y} - k_1 y + MA\Omega^2 \sin(\Omega t), & y \leq d \\ -R_1\dot{y} - (k_1 + k_2)y + MA\Omega^2 \sin(\Omega t) + k_2 d, & y > d \end{cases} \tag{1}$$

The non-dimensionalised equations of motion for the bilinear oscillator is given by Eq. 2

$$\ddot{x} + 2\zeta\dot{x} + x + \beta(x - e)H(x - e) = F_0 \sin(\omega\tau), \tag{2}$$

where $x = y/y_0$ is the non-dimensionalised displacement, $\dot{x}$ is the non-dimensionalised velocity, $\tau = \omega_n t$ is the non-dimensional time, $\beta = k_2/k_1$ is the stiffness ratio, $e = d/x_0$ is the non-dimensional gap, F_0 is the non-dimensional forcing amplitude, $\zeta = c/2m\omega_n$ is the damping ratio and the over-dot represents the derivative with respect to τ. $H(.)$ represents the Heaviside step function that demarcates the region starting where, the mass M and secondary stiffness K_2 are in contact. A simplifying assumption that the discontinuity surface is not in motion and is located at $x - e = 0$, has been made. The swiching between the two state spaces, thus always happens at $x = e$, making the analyses easier.

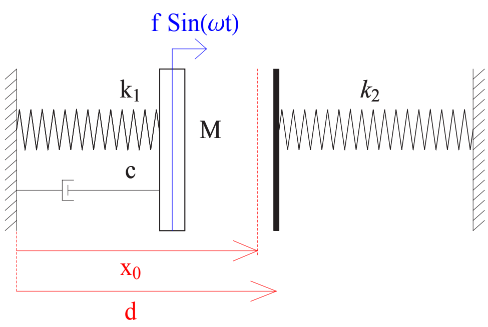


Figure 1: Schematic of the impact oscillator

The system of equations were solved using point-wise mapping technique using 4^{th} order Runge-Kutta method. For the value of parameters defined as $\zeta = 0.01, \beta = 29, e = 1.26, m = 2.1, \omega = 2/m, a = 0.75, f = a\omega^2$, two co-existing stable solutions of periodicity one and three were noted as seen in Fig. 2.

The group of fundamental periodic motions were characterized by the different number of impacts in one motion period, which equals the excitation force period. The Poincaré sections were taken stroboscopically *i.e.*, each at a time period of $2\pi/\omega$. These are shown as blue circles in Figs 2(a), 2(b). It can also be confirmed with the corresponding bifurcation plot and basin of attraction that no other stable attractors exist for the regime being taken into study.

The explanation for other parameters that were chosen is as follows. It is known from [9] that in impacting mechanical systems large-amplitude chaotic oscillations develop where collisions with the secondary stiffness are

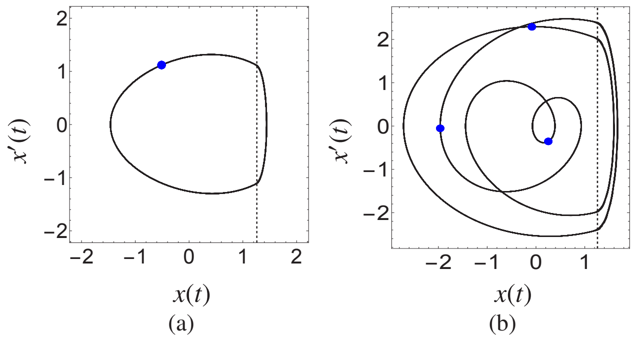


Figure 2: (a)Phase portrait and Poincaré section for period 1 oscillations, (b) period 3 oscillations; the blue points correspond to the Poincaré points.

with near-zero velocity, *i.e.* when grazing is encountered. These oscillations vanish when the forcing frequency is an integer multiple or is nearly equal to an integer multiple of the natural frequency of the system. Here $\omega_n = 1$, hence our choice of $\omega = 0.952$ (obtained as the ratio of frequencies $m = 2\omega_n/\omega$ is taken to be 2.1) happens to evade the condition where chaotic orbits would be encountered, thus simplifying the analyses of the effect of noise perturbations. The parameter of choice avoids the condition of grazing, hence chaotic orbits are not created for the domain being studied.

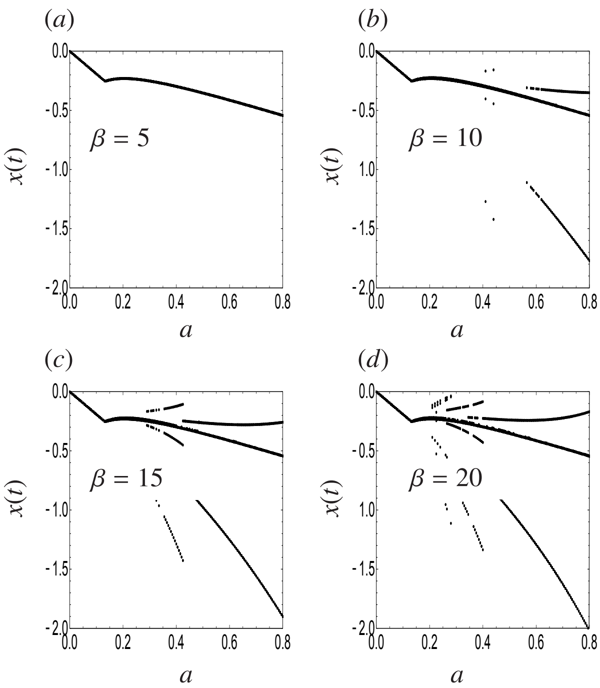


Figure 3: Figure showing how co-existing attractors come to originate as the secondary subspace gets stiffer.

Fig. 3 shows the bifurcation diagram with a for various values of β. Fig 3 (a) shows a single curve illustrating that only one stable solution exists for $\beta = 5$. As β is increased, one can see the emergence of multiple lines indicating the presence of coexisting attractors. These other attractors appear as a consequence of unstable periodic orbits born due to hitting at the secondary boundary, which become stable as a increases. Further increase of β ensures that the unstable periodic orbits gain stability at smaller values of a, thereby not creating unstable periodic or chaotic saddles near the regime being studied. Thus, a significantly higher value of β, *i.e.*,29 was chosen. The results of the brute force bifurcation plots showing stable sets are available in [9].

3 Introduction of noise

A random perturbation over the sinusoidally varying forcing was input to the system. The forcing F(t) now consists of a harmonic function superimposed by small amplitude noise modelled as an Ornstein Uhlenbeck process and is mathematically expressed as:

$$F(t) = F_0 \sin(\omega\tau) + n\lambda, \qquad (3)$$

where n is the noise level of the parametric fluctuations and $\lambda(t)$ is a sample realisation of the Ornstein-Uhlenbeck(OU) process with process parameters mean, volatility, and mean reversion speed as 0, 0.1 and 0.3 respectively. The noise was added at each step of the carried out fixed-step Runge-Kutta integration. The input λ to the forcing was made to take in a new value at every new step of integration performed. This led to a deviation from the deterministic behaviour of the system. The OU process has been chosen as it is the only Gauss–Markov process with a bounded variance that admits a stationary probability distribution. It thus facilitates the study of the effect of correlation lengths of the noise input on the behaviour of the system. The OU process has been deployed to study the effect of over-damped linear springs in the presence of random fluctuations and thus very well suits the model being studied [10]. Over time, the process tends to drift towards its long-term mean. Such a process is called mean-reverting. Hence, the mean reversion speed is an important parameter of the Ornstein-Uhlenbeck process. The autocorrelation for the process is given by $\exp[\theta|\tau|]\sigma^2/2\theta$. When $n = 0.385f$, the percentage of noise, with respect to the deterministic signal, that is input to the system is $0.107 \pm 3.45\%$ and the forcing input is as shown in Fig. 4. The forcing thus becomes amplitude modulated, the modulation happening at random, as it still retains its sinusoidal character.

The governing equation was integrated for 10^4s. The phase plot diagram obtained from the numerically integrated trajectories of the field equations are shown in Fig 5. These phase diagrams were plotted with trajectories from $(0.45 - 0.75) \times 10^4$ s and $(0.8 - 1) \times 10^4$ s. These figures are superimposed with the phase plot trajectories corresponding to period 1 (red line) and period 3 (green line) oscillations for the corresponding deterministicially excited system. Additionally, the Poincare points obtained by sampling at period of $2\pi/\omega$ are also shown in these figures as blue points. It is seen that these blue points form a cluster in the vicinity where the Poincare points were

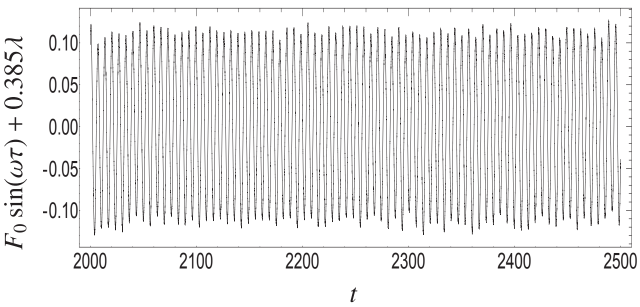
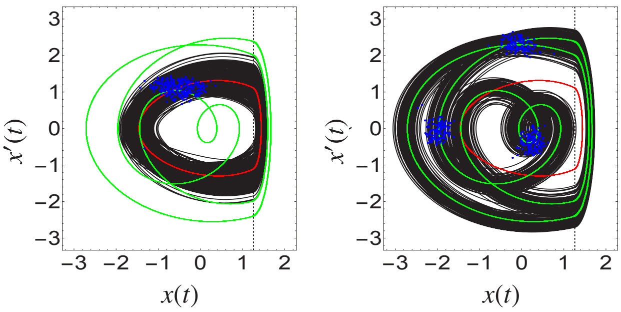


Figure 4: Noisy sinusoidal forcing to the vibroimpact system.



(a) (b)

Figure 5: (a) Phase portrait and Poincaré section for time $(0.8 - 1)10^4$. The response of random input seen to follow modulated period 1 oscillations. (b) Phase portrait and Poincaré section for time $(0.45 - 0.75)10^4$. The response of random input seen to follow modulated period 3 oscillations.

observed in the phase space diagram corresponding to the deterministic system; see Fig 5.

Thus, it is observed that the system's long term behavior fluctuates from a modulated period 3 motion to a modulated period 1 motion. The corresponding time histories presented in Fig. 7 confirm that the response of the system flips between two qualitatively distinct dynamics. Such irregular alternation between qualitatively distinct dynamics in the presence of noise is known as noise-induced intermittency in nonlinear dynamical systems literature, see [8]. This explanation for why such a flipping behavior is observed can be provided with the help of bifurcation diagrams constructed in the presence of noise. Such bifurcation diagrams constructed give an idea of the spread of the modulated periodic attractor [11].

When $n \leq 0.385f$, in the bifurcation diagram at $a = 0.75$, the red regions, corresponding to the bifurcation points in the presence of noise, following period 3 and period 1 branches do not overlap each other. However, as the intensity of noise is increased, the regions begin to overlap, causing the sytem to exhibit such switching back and forth between two different dynamics as is seen in Fig. 6. The deterministic bifurcation diagram, in black, shows four arms at $a = 0.75$ corresponding to three arms of a period 3 cycle at $x = -1.962, -0.083, 0.25$ and one arm

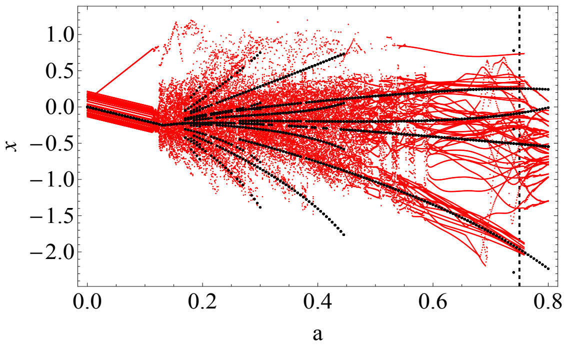


Figure 6: Bifurcation diagram of the bilinear oscillator system in the presence of noise (in red) of intensity $n = 0.385f$ superimposed over the deterministic bifurcation plot (in black).

of a co-existing period 1 invariant set at $x = -0.511$. The red regions in the bifurcation diagram correspond to the stroboscopically recorded bifurcation points in the noisy case. The corresponding time histories for different noise intensities are as seen in Fig. 7. It has been observed that increasing the noise levels leads to more increase in the basin hopping phenomenon. The higher amplitude period 3 and lower amplitude period 1 oscillations are observed to persist for significant amounts of time until the external forcing makes the system change its behavior. The trajectory takes turns to visit, by sudden bursts, to the system's two coexisting attractors.

To further investigate the effects of a parametric noise perturbation on the final structure of the basins of the co-existing attractors, a two-dimensional section of the phase space $(x_0; x'_0)$ with $(x_0; x'_0)$ as initial conditions were taken and evolved for a time of $10^4 s$. The first 300 cycles were dropped off and the responses of the system in the next 1400 cycles were analyzed in a grid of 160×160 pixels and the final basin structures were drawn.

In order to draw the basins, the spread of the Poincaré points were studied. A small elliptical region with major and minor axes taken along confidence lines of σ around x and $\dot{x}$ about the four deterministic Poincaré points. These ellipses were then considered as thresholds for the recorded modulated period 1 and period 3 responses. Thus, if $> 66.67\%$ of the points happened to congregate around the period 1 Poincaré point and almost none fell into the regions of period 3 points, say, then the behaviour was defined to be modulated period 1 throughout. For a case where all the regions around period 1 and period 3 were visited and the duration the system spent exhibiting transitory behaviour, given by the number of points outside the thresholds, was less in comparision to what it spent in the basins of the co-existing invariant sets, the behavior was considered to be intermittent. Thus, if the outliers from both the regions were within a certain number, which here is taken as $1/4^{th}$ of the total number of points, then the system was defined to be undergoing intermittent oscillations. Otherwise the system was said to undergo random oscillations. The following color scheme has been adopted in Fig 8: (a) if the behavior is modulated period 1, it is marked as red, (b) if the behavior is modulated period 3, it is marked as blue, (c) if the system is understood to be visiting both the attractors and the system has quick transition times *i.e.* if less points fall outside the designated domains, it is marked magenta, (d) if the system is understood to be visiting both the attractors and more points fall outside the designated domains, it is understood that the system exhibits random oscillations and is marked in yellow.

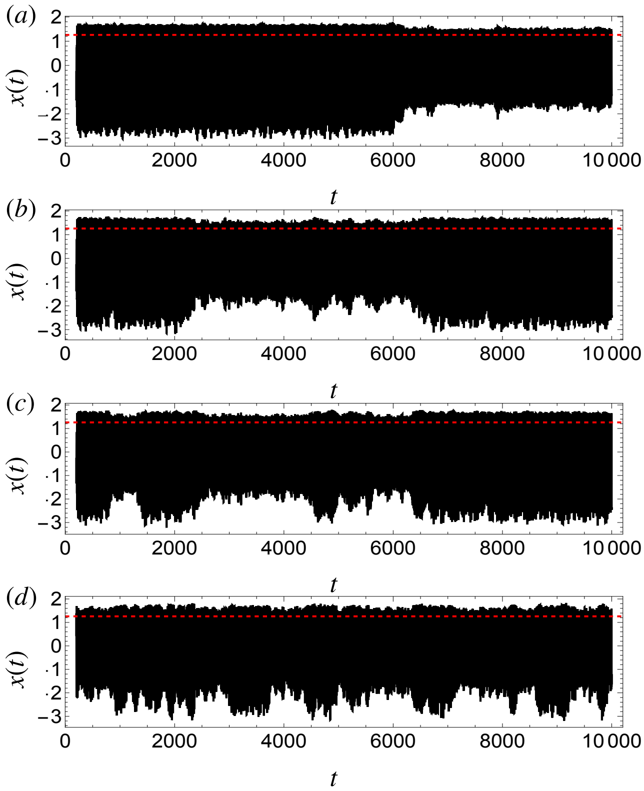


Figure 7: Time histories showing flipping behavior when (a) $n = 0.385f$, (b) $n = 0.6f$, (c) $n = 0.7f$, (d) $n = 0.9f$. The red line indicates the presence of the discontinuity boundary.

It is seen in Fig. 8 that when the amplitude of noise is $n = 0.385f$, the intermittent regions are barely existent. As the forcing input signal gets noisier, the regions of modulated throughout-homogeneous periodic structures shrink until they are completely lost to irregular or random oscillations. The basin structures are however seen to have been retained but the basin boundaries get smeared out as the noise intensifies.

4 Conclusion

In this paper it has been shown that the parametric noise in the forcing input leads the system to hop between the stable co-existing periodic attractors of the bilinear oscillator system, even when the basins of attraction have well defined and significant basin volumes for each of the present attractors. It has been shown that the long term behavior of the system might alternate between the possible distinct stable states. An explanation for basin hopping

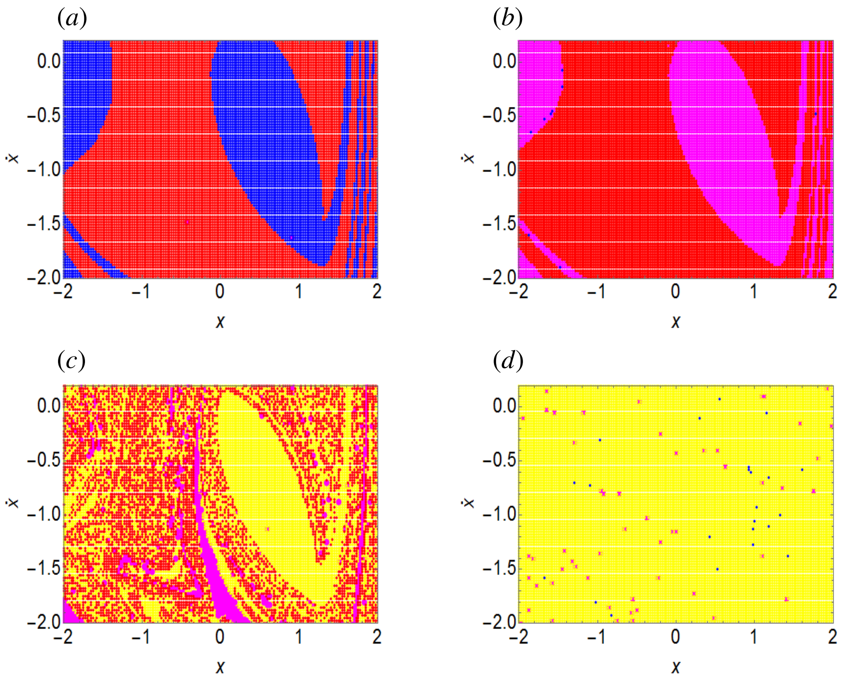


Figure 8: Basins of attraction drawn for 1400 cycles beyond transience for noise levels (a) $n = 0.385f$, (b) $n = 0.5f$, (c) $n = 0.6f$, (d) $n = 0.8f$. The colours red, blue, magenta and yellow correspond to initial conditions that display modulated period 1 motion, modulated period 3 motion, intermittent flipping and random motions respectively.

was described by [12] to be due to the properties of the non attracting stable states, also known as saddles, lying in between the basins. The trajectory, while wandering in the vicinity of such saddles, experience a transient irregular motion. The noise term pushes the trajectory out of an open neighborhood of some attractor into the basin boundary. The trajectory spends there a certain amount of time until it reaches again the neighborhood of the same or another attractor[4]. This study can be very complex for structures where many stable and chaotic states co-exist. More studies are needed to be carried out to study such scenarios.

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