Some considerations on modelling and homogenization of multiphase light alloys

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Abstract. The modelling strategies aimed to access the thermo-mechanical behavior of the up-to date light and superlight alloys should take into account characteristics of each particular manufacturing technology. In the present study the light alloys and materials of high porosity are regarded as multiphase composites for which the content of embedded phases is predominant. A new modification of differential effective medium theory is presented accounting for inclusion's size effects. The applied approach is based on the size-sensitive dilute homogenization procedure with Cosserat material matrix. Numerical examples for elastic properties of a close cell foam materials with different Poisson's ratio are presented and discussed.

1 Introduction

In recent years, the development of enhanced technologies for preparing of composites with higher volume fractions of secondary phases shows a significant increase. Special attention has been attracted to light alloys and other materials of metal or ceramic matrix [1] , [2] , [3]. Metal foams are materials of high porosity which possess specific combination of thermo-mechanical and physical properties, [4]. For their very low specific weights and thus high specific stiffnesses, they are able to absorb significant amount of deformation energy while guaranteeing other properties such as high fire and heat resistance, noise attenuation and shielding of electromagnetic devices. The unique combination of useful characteristics makes the application of metal foams an rising industry in many engineering fields from interior design and equipment to civil engineering and vehicle construction, see [5].

There exist several approaches to obtain porosity – properties relations for such composites. Among them one could differentiate the methods of micro-mechanics as differential [6], [7], [8] and self-consistent methods [9], FEM [10], minimum solid area models [11] , as well as commonly used semi empirical models [12], [13], [14], [15], and etc. Recently a comparative study has been published in [5] covering most elastic modulus–porosity relationships suggested in the literature.

In [10] Garbozci and Berryman presented a variant of Differential Effective Medium (DEM) theory where the size effect is taken into account introducing a composite particle (an inner particle surrounded by a thin shell with different properties). This new slightly larger inclusion is firstly homogenized following Christensen's method, [9]. In the mentioned approach the size sensitivity of the model has been expressed via the ratio between the diameters of the inner particle and outer spherical shell. In such way the size effect vanishes if the thickness of the shell tends to zero. From our point

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of view this variant can't be recommended for foams and other composites of high porosity. The volume fracture of the shell is comparable with matrix volume fraction and it is very hard to avoid the overlapping among homogenized effective inclusions. Besides, the asumption that the shell of dense material being obviously a part of the matrix around the inclusion should be incorporated to a void, is open to criticism too.

This paper is aimed to develop a variant of DEM theory accounting for size effects in multiphase composites with minor matrix fraction, including closed cells foams. Approximate analytical formulae are derived in order to assess and analyze the influence of pore size on the elastic properties of porous composite with higher porosity. The homogenization technique based on DEM theory and modified Mori - Tanaka two-phase model [16],[17] for the case of spherical inclusions is further developed incorporating an additional size dependent term to one of the classical Eshelby's tensor components. The consequences of this new assumption are demonstrated on a few limiting cases. For them the system of differential equations is explicitly solved and numerical computations elucidate the influence of pores-matrix microstructure interaction on overall elastic moduli.

2 Modelling

For obtaining the overall properties of a multiphase composite at higher concentration of embedded phases two–steps homogenization procedure is suggested. On the first step the Representative Volume Element (RVE) of the material is distributed into pseudo grains, following the ideas from [18]. According to this approach the multiphase RVE, consisting of matrix and n_f phases, is equivalent to a RVE, consisting of n pseudo grains, $n \geq n_f$. Each pseudo grain is a two-phase composite, containing a part of the matrix and all inclusions with a particular size and elastic properties. Each pseudo grain should be homogenized as a two phase composite following a proper scheme which depends on the total volume fraction of the inclusions C_{sum}. For composites with low volume fraction of matrix phase as closed cell foams or porous metals a new variant of DEM is presented and discussed in this paper. The proper choice of a homogenization method at the fist step is a key point for the overall properties predictions of entire model. On the second step the RVE's agglomerate of the already homogeneous Cauchy -type pseudo-grains has to be subjected to the final homogenization performing Hill-Budiansky scheme. For a composite with known size distribution function of inclusions the variant developed in [19] is appropriate at the second step.

2.1 Limits of size effect

In the case of low volume fraction, not exceeding 20-30 %, the hypothesis of dilute inclusions is valid and updated size sensitive Mori-Tanaka homogenization could be applied [16],[19]. For the RVE of the composite with inclusions entirely surrounded by a matrix the moduli of $i-th$ pseudo-grain are obtained by means of equations:

$$K_{ci} = K_m + \frac{C_{sum}K_m\left(K_i - K_m\right)}{C_m a_m\left(K_i - K_m\right) + K_m},\; G_{ci} = G_m + \frac{C_{sum}G_m\left(G_i - G_m\right)}{C_m\left(b_m - b_{0i}\right)\left(G_i - G_m\right) + G_m} \tag{1}$$

where: $a_m = \frac{3K_m}{3K_m+4G_m}$, $b_m = \frac{6(K_m+2G_m)}{5(3K_m+4G_m)}$. According to the variant of Cosserat theory adopted, [17] , the following relations holds among the parameters of initially micropolar matrix:

$$(l_0)^2 = \frac{\beta}{G_0} = \frac{\gamma}{\kappa} \quad , \quad \frac{\kappa}{G_0} = \frac{\gamma}{\beta} = p \tag{2}$$

In (2) the moduli G_0 and β connect symmetric parts of stresses - strains, and couple stress- curvature tensors, respectively, and κ and γ connect antisymmetric parts of corresponding tensors at elastic state of the initial micropolar matrix, $p \geq 0$. The size sensitivity of the model depends on the dimensionless parameter D_i/l_0 through the average Eshelby tensor component $(b_m - b_{0i})$, [16],[17], where l_0 is intrinsic length of the initial matrix and D_i is inclusion's diameter.

$$
\begin{gathered}
b_{0i} = \tfrac{6p}{5(p+1)} R_i(\eta_i) \ , \\
R_i(\eta_i) = e^{-\eta_i}\left(\eta_i^{-2} + \eta_i^{-3}\right)(\eta_i ch\eta_i - sh\eta_i) \ , \\
\eta_i = \tfrac{D_i}{2h} \ , \quad h = \tfrac{p+1}{\sqrt{p}}\tfrac{l_0}{2} \ , \quad \eta_i = \tfrac{\sqrt{p}}{p+1}\tfrac{D_i}{l_0}
\end{gathered}
\tag{3}
$$

It is important to emphasize that after the procedure, described by (1) the homogenized material is isotropic Cauchy - type medium and the influence of the initial matrix as centrosymmetric micropolar material is explicitly included in the term b_{0i}. Now for the case of spherical pores (closed cells) limiting values, which serve as bonds for assessment of the size effects will be presented. At the beginning appling some algebra it is proven that the function $R_i(\eta_i)$ behaves as demonstrated on Fig.1.

$$
R_i\big|_{\eta_i \to 0} = 1/3 \quad , \quad R_i\big|_{\eta_i \to \infty} = 0 \tag{4}
$$

As far as $\eta_i \geq 0$ the maximum size effect is expected when η_i tends to zero.

Case I $\quad p = 0, \quad (\kappa = 0)$

$$
\begin{aligned}
\eta_i &= 0 \quad , \quad R_i(0) = 1/3 \\
b_{0i} &= 0
\end{aligned}
\tag{5}
$$

In this case there is no size effect and results coincide with the classical Mori-Tanaka ones.

Case II $\quad p = 1, \quad (\kappa = G_0)$

$$
\begin{aligned}
\eta_i &= \frac{D_i}{2l_0} \quad , \quad 0 < R_i = R_i\left(\frac{D_i}{2l_0}\right) \leqq 1/3 \\
b_{0i} &= \frac{3}{5} R_i\left(\frac{D_i}{2l_0}\right) \quad , \quad 0 < b_{0i} \leqq 1/5
\end{aligned}
\tag{6}
$$

Herein it is assumed (as in many investigations [16], [17],[20] concerning the application of centrosymmetric micropolar theory) that the Cosserat parameter κ is in order of the shear modulus of initial matrix. If the inclusions (pores) are much smaller than the characteristic length l_0, the term b_{0i} converges to $1/5$.

Case III $\quad p = \infty, \quad (\kappa = \infty)$;

$$
\begin{aligned}
\lim \eta_i\,|_{\kappa \to \infty} &= 0 \quad , \quad R_i(0) = 1/3 \\
\lim b_{0i}\,|_{\kappa \to \infty} &= 2/5
\end{aligned}
\tag{7}
$$

Case I and Case III trace out the bounds from minimum to maximum susceptibility of the matrix to the inclusions fitting in it.

2.2 Differential approach

The differential effective medium approach requires an explicit homogenization scheme for a dilute inclusion problem to be postulated at the beginning. Let the equations (8) represent such a model for the bulk and shear modulus of composite, respectively:

$$K = F_K\left(K_m, K_i, c, G_m, G_i\right) \quad , \quad G = F_G\left(K_m, K_i, c, G_m, G_i\right) \tag{8}$$

It is supposed that functions F_K and F_G satisfy the requirements of Taylor's theorem and are expanded in Maclaurin series with respect of concentration c. Then:

$$\begin{aligned} K &= F_K(0) + \frac{dF_K}{dc}(0)\,c + O_K\left(c^2\right) , \\ G &= F_G(0) + \frac{dF_G}{dc}(0)\,c + O_G\left(c^2\right) \end{aligned} \tag{9}$$

where $F_K(0) = K_m$ and $F_G(0) = G_m$.

The linear parts of (9) as dilute relations are used now to generate approximate differential equations suitable to estimate the elastic moduli of the composite when arbitrary amount of the included phase is embedded into the matrix, [10]. Suppose that RVE of the composite material V contains c volume fraction of inclusions and $\phi = (1 - c)$ volume fraction of matrix. Let one remove from the RVE a differential volume element dV. In this volume element the matrix part is $dV_m = (1 - c))\,dV$ and after that a new portion of the inclusions is added to the composite replacing the differential volume element with the same small volume consisting of inclusions only. Than, the volume fraction of the matrix will change as follows:

$$\phi + d\phi = \frac{V_m - dV_m}{V - dV + dV} = \phi - (1 - c)\frac{dV}{V} \Longrightarrow \frac{dc}{1 - c} = \frac{dV}{V} . \tag{10}$$

The main idea of the DEM is to consider the new inclusion enriched material as a composite consisting of matrix of equivalent material and second phase of inclusions with a very low volume fraction dV/V. For such composite the dilute hypothesis holds and the overall elastic properties can be obtained by simplified linear formulae (9). The linear system (9) is rewritten in the following form: the moduli K, G are expressed as $\left(K_{eq} + dK_{eq}\right), \left(G_{eq} + dG_{eq}\right)$ and the matrix index "m" is replaced by matrix index "eq" (equivalent):

$$\begin{aligned} K_{eq} + dK_{eq} &= K_{eq} + F_K{}'(eqv, 0)\frac{dV}{V} , \\ G_{eq} + dG_{eq} &= G_{eq} + F_G{}'(eqv, 0)\frac{dV}{V} ; \end{aligned} \tag{11}$$

$$\begin{aligned} dK_{eq} &= \frac{dF_K\left(K_{eq}, K_i, G_{eq}, G_i\right)}{dc}\Big|_{c=o}\frac{dc}{1 - c} , \\ dG_{eq} &= \frac{dF_G\left(K_{eq}, K_i, G_{eq}, G_i\right)}{dc}\Big|_{c=o}\frac{dc}{1 - c} . \end{aligned} \tag{12}$$

2.3 Size - sensitive differential approach

It is worth noting that the system of differential equations (12) can be obtained for every dilute homogenization (8) which allows Taylor series expansion to be performed. Further the homogenization scheme (1) is used as a representative of (8) , see [17], [16], leading to the system:

$$\begin{aligned} dK_{eq} &= \frac{\left(3K_{eq} + 4G_{eq}\right)\left(K_i - K_{eq}\right)}{\left(3K_i + 4G_{eq}\right)}\frac{dc}{1 - c} , \\ dG_{eq} &= \frac{G_{eq}\left(G_i - G_{eq}\right)}{\left(G_i - G_{eq}\right)\left[\frac{5}{6}\frac{\left(K_{eq}+2G_{eq}\right)}{\left(3K_{eq}+4G_{eq}\right)} - b_{0i}\right]}\frac{dc}{1 - c} . \end{aligned} \tag{13}$$

The term b_{0i} appearing in (3) and (13) is the only trace of Cosserat properties of the initial matrix in the average Eshelby tensor. The new assumption addopted herein is that the term b_{0i} could be regarded as an internal model parameter which expresses in average maner the sensitivity of the genuine matrix to the presence of spherical inclusions of diameter D_i. It depends on dimentionless model parameters p and D_i/l_0. For metal based composites internal lenght l_0 is often related to the grain size, i.e. to the matrix microstructure self-organization. From the DEM's point of view such assuption seems resonable since DEM supposes that accounting for addition of another small portion of inclusions the simple linear homogenization procedure (11) should be executed over and over again. Every time the eqivalent material, obtained at the previous step has to play the roll of matrix at the next step. As a complementary agrument one may point out that after such kind of homogenization (1) the overall modulae of the composite define an ordinary isotropic Cauchy - tipe material without any micropolar features.

2.3.1 Size - sensitive DEM theory in case of porous inclusions

In the case of spherical pores inclusions or closed cells foams the system of differential equations (13) expressed in terms of Poisson's ratio and Young's modulus can be uncoupled. To this end we consider the moduli ν_{eq} and E_{eq} obey the common relations $\nu_{eq} = \frac{(3K_{eq}-2G_{eq})}{2(G_{eq}+3K_{eq})}$, $E_{eq} = \frac{9K_{eq}G_{eq}}{(G_{eq}+3K_{eq})}$, $\frac{G_{eq}}{K_{eq}} = \frac{3(1-2\nu_{eq})}{2(1+\nu_{eq})}$ as $\nu_m = \frac{(3K_m-2G_m)}{2(G_m+3K_m)}$, $E_m = \frac{9K_mG_m}{(G_m+3K_m)}$, $\frac{G_m}{K_m} = \frac{3(1-2\nu_m)}{2(1+\nu_m)}$. Then the system (13) is transformed into the equations:

$$\frac{2}{3}\frac{\left(7-5\nu_{eq}\right)+15\left(1-\nu_{eq}\right)b_{0i}}{\left(1-\nu_{eq}{}^2\right)\left[\left(1-5\nu_{eq}\right)-5\left(1-\nu_{eq}\right)b_{0i}\right]}d\nu_{eq} = \frac{dc}{1-c}\ , \tag{14}$$

$$\frac{dE_{eq}}{E_{eq}} = \frac{1}{2\left(2\nu_{eq}-1\right)}\left[3\left(1-\nu_{eq}\right)+4F_\nu{}'\left(eqv,0\right)\right]\frac{dc}{1-c}\ . \tag{15}$$

where:

$$F_\nu{}'\left(eqv,0\right) = \frac{3}{2}\frac{\left(1-\nu_{eq}{}^2\right)\left[\left(1-5\nu_{eq}\right)-5\left(1-\nu_{eq}\right)b_{0i}\right]}{\left(7-5\nu_{eq}\right)+15\left(1-\nu_{eq}\right)b_{0i}}\ . \tag{16}$$

Taking into account that $F_\nu{}'\left(eqv,0\right)\frac{dc}{1-c} = d\nu_{eq}$ and if $F_\nu{}'\left(eqv,0\right) \neq 0$ the equation (15) can be now represented by the varables E_{eq} and ν_{eq} :

$$\frac{dE_{eq}}{E_{eq}} = \frac{1}{2\left(2\nu_{eq}-1\right)}\left[\frac{(3\left(1-\nu_{eq}\right)}{F_\nu{}'\left(eqv,0\right)}+4\right]d\nu_{eq}\ . \tag{17}$$

$$F_\nu{}'\left(eqv,0\right) = 0 \quad \Leftrightarrow 1-5\nu_{eq}-5\left(1-\nu_{eq}\right)b_{0i} = 0 \tag{18}$$

The condition (18) gives particular solutions of the equations (14) and (15), respectively. For any admissible parameter $0 \leqq b_{0i} \leqq 2/5$ there exists a single value $\nu_m{}^*$ satisfying the condition (18) so that:

$$\nu_m{}^* = \left(1-5b_{0i}\right)/5\left(1-b_{0i}\right) \tag{19}$$

and eqns. (21) are solutions of the system (20):

$$d\nu_{eq} = 0 \quad ; \quad \frac{dE_{eq}}{E_{eq}} = \frac{3\left(1-\nu_m{}^*\right)}{2\left(2\nu_m{}^*-1\right)}\frac{dc}{1-c}\ . \tag{20}$$

$$\nu_{eq}(c) = \nu_m{}^* = const \quad ; \quad E_{eq}(c, \nu_m{}^*) = E_m (1-c)^{\frac{3(1-\nu_m{}^*)}{2(1-2\nu_m{}^*)}} \quad . \tag{21}$$

The bulk and shear moduli of the composite corresponding to (21) are:

$$K_{eq}(c, \nu_m{}^*) = \frac{E_m (1-c)^{\frac{3(1-\nu_m{}^*)}{2(1-2\nu_m{}^*)}}}{3(1-2\nu_m{}^*)} \quad ; \quad G_{eq}(c, \nu_m{}^*) = \frac{E_m (1-c)^{\frac{3(1-\nu_m{}^*)}{2(1-2\nu_m{}^*)}}}{2(1+\nu_m{}^*)} \quad . \tag{22}$$

3 Results and discussions

Differential equations (14) and (17) should be solved consecutively one after another. For some special values of b_{0i} the analytical solutions are presented in the next sections.

3.1 Size - sensitive DEM theory in case I

Case I $\quad b_{0i} = 0, \quad (p = 0)$. Differential equations (14) and (17) take now the form (23), (24):

$$\frac{2}{3} \frac{\left(7 - 5\nu_{eq}\right)}{\left(1 - \nu_{eq}{}^2\right)\left(1 - 5\nu_{eq}\right)} d\nu_{eq} = \frac{dc}{1-c} \ , \tag{23}$$

$$\frac{dE_{eq}}{E_{eq}} = \frac{1}{\left(1 - 2\nu_{eq}\right)} \left[\frac{\left(5\nu_{eq} - 7\right)}{\left(1 + \nu_{eq}\right)\left(1 - 5\nu_{eq}\right)} - 2 \right] d\nu_{eq} \ ; \tag{24}$$

with solutions:

$$\left(\frac{1-\nu_m}{1-\nu_{eq}}\right)^{1/6} \left(\frac{1+\nu_m}{1+\nu_{eq}}\right)^{2/3} \left(\frac{1-5\nu_{eq}}{1-5\nu_m}\right)^{5/6} = 1 - c \tag{25}$$

$$E_{eq} = E_m \left(\frac{1+\nu_m}{1+\nu_{eq}}\right)^{2/3} \left(\frac{1-5\nu_{eq}}{1-5\nu_m}\right)^{5/3} \tag{26}$$

$$\nu_m{}^* = 1/5 \quad ; \quad E_{eq}(c, \nu_m{}^*) = E_m (1-c)^2 \tag{27}$$

$$K_{eq}(c, \nu_m{}^*) = \frac{5}{9} E_m (1-c)^2 \quad ; \quad G_{eq}(c, \nu_m{}^*) = \frac{5}{12} E_m (1-c)^2 \ . \tag{28}$$

Case I describes those type of matrix –inclusions interactions when embedded phase has much bigger size than the internal parameter l_0 dealing with matrix microstructure. In other words the microstructure of the matrix is so fine, that the matrix does not "feel" the presence of the coarse added phase. Of course, case I also represents the classical solution of DEM when size effects are not considered at all, see [8].

3.2 Size - sensitive DEM theory in case II

Case II $\quad b_{0i} = 1/5, \quad (p = 1)$. Differential equations (14) and (17) take now the form (29), (30):

$$-\frac{3}{2} \frac{\left(13 - 11\nu_{eq}\right)}{\left(1 - \nu_{eq}{}^2\right)\nu_{eq}} d\nu_{eq} = \frac{dc}{1-c} \ , \tag{29}$$

$$\frac{dE_{eq}}{E_{eq}} = \frac{1}{\left(1 - 2\nu_{eq}\right)} \left[\frac{\left(5 - 4\nu_{eq}\right)}{2\left(1 + \nu_{eq}\right)\nu_{eq}} - 2 \right] d\nu_{eq} \ ; \tag{30}$$

and the solutions are:

$$\left(\frac{1-\nu_m}{1-\nu_{eq}}\right)^{1/6}\left(\frac{1+\nu_m}{1+\nu_{eq}}\right)^{3/2}\left(\frac{\nu_{eq}}{\nu_m}\right)^{5/3}=1-c \tag{31}$$

$$E_{eq}=E_m\left(\frac{1+\nu_m}{1+\nu_{eq}}\right)^{3/2}\left(\frac{\nu_{eq}}{\nu_m}\right)^{5/2} \tag{32}$$

$$\nu_m{}^*=0 \quad ; \quad E_{eq}\left(c,\nu_m{}^*\right)=E_m\left(1-c\right)^{3/2} \tag{33}$$

$$K_{eq}\left(c,\nu_m{}^*\right)=\frac{1}{3}E_m\left(1-c\right)^{3/2} \quad ; \quad G_{eq}\left(c,\nu_m{}^*\right)=\frac{1}{2}E_m\left(1-c\right)^{3/2}\ . \tag{34}$$

Case II in this subsection is a special variant of the case II described in the section 2.1. It represents the limiting case when the diameter of inclusions tends to zero and Cauchy initial shear modulus G_0 is of the same order than Cosserat modulus κ. Expressed by b_{0i} this case is situated just in the middle between Case I and Case III.

3.3 Size - sensitive DEM theory in case III

Case III $b_{0i}=2/5$, $(p=\infty)$. Differential equations (14) and (17) take now the form (35), (36):

$$-\frac{2}{3}\frac{\left(13-11\nu_{eq}\right)}{\left(1-\nu_{eq}{}^2\right)\left(1+3\nu_{eq}\right)}d\nu_{eq}=\frac{dc}{1-c}\ , \tag{35}$$

$$\frac{dE_{eq}}{E_{eq}}=\frac{1}{\left(1-2\nu_{eq}\right)}\left[\frac{\left(13-11\nu_{eq}\right)}{\left(1+\nu_{eq}\right)\left(1+3\nu_{eq}\right)}+2\right]d\nu_{eq}\ ; \tag{36}$$

with solutions given by:

$$\left(\frac{1-\nu_m}{1-\nu_{eq}}\right)^{1/6}\left(\frac{1+\nu_m}{1+\nu_{eq}}\right)^{4}\left(\frac{1+3\nu_{eq}}{1+3\nu_m}\right)^{25/6}=1-c \tag{37}$$

$$E_{eq}=E_m\left(\frac{1+\nu_m}{1+\nu_{eq}}\right)^{4}\left(\frac{1+3\nu_{eq}}{1+3\nu_m}\right)^{5} \tag{38}$$

$$\nu_m{}^*=-1/3 \quad ; \quad E_{eq}\left(c,\nu_m{}^*\right)=E_m\left(1-c\right)^{6/5} \tag{39}$$

$$K_{eq}\left(c,\nu_m{}^*\right)=\frac{1}{5}E_m\left(1-c\right)^{6/5} \quad ; \quad G_{eq}\left(c,\nu_m{}^*\right)=\frac{3}{4}E_m\left(1-c\right)^{6/5}\ . \tag{40}$$

If one remember that for any anisotropic material Poisson's ratio should lie in the interval $-1<\nu<1/2$, providing positive values of the other Cauchy elastic moduli, the current value $\nu_m{}^*=-1/3$ is admissible and corresponds to a material of auxetic type, see [21], [4]. Theoretically at $p\to\infty$ i.e. at such ratio between mentioned Cauchy and Cosserat moduli the material of the initial matrix is the most sensitive to the presence of inclusions. This case also corresponds to the variant when $G_0\ll\kappa$. Case III manifests the limited "resistance" of the matrix when very small extraneous objects give rise to obstacles blocking the matrix to develop its own structures with size $l_0>>D_i$.

3.4 Size - sensitive DEM theory in case IV

Case IV $b_{0i} = 1/10$, $(p = 1)$. Differential equations (14) and (17) take now the form (41), (42):

$$\frac{2}{3}\frac{\left(17-13\nu_{eq}\right)}{\left(1-\nu_{eq}{}^{2}\right)\left(1-9\nu_{eq}\right)}d\nu_{eq}=\frac{dc}{1-c}\ , \tag{41}$$

$$\frac{dE_{eq}}{E_{eq}}=\frac{1}{\left(1-2\nu_{eq}\right)}\left[\frac{\left(13\nu_{eq}-17\right)}{\left(1+\nu_{eq}\right)\left(1-9\nu_{eq}\right)}-2\right]d\nu_{eq}\ . \tag{42}$$

Derived solutions are as follows:

$$\left(\frac{1-\nu_m}{1-\nu_{eq}}\right)^{1/6}\left(\frac{1+\nu_m}{1+\nu_{eq}}\right)\left(\frac{1-9\nu_{eq}}{1-9\nu_m}\right)^{7/6}=1-c \tag{43}$$

$$E_{eq}=E_m\left(\frac{1+\nu_m}{1+\nu_{eq}}\right)\left(\frac{1-9\nu_{eq}}{1-9\nu_m}\right)^{2} \tag{44}$$

$$\nu_m{}^{*}=1/9 \quad ; \quad E_{eq}\left(c,\nu_m{}^{*}\right)=E_m\left(1-c\right)^{12/7} \tag{45}$$

$$K_{eq}\left(c,\nu_m{}^{*}\right)=\frac{3}{7}E_m\left(1-c\right)^{12/7} \quad ; \quad G_{eq}\left(c,\nu_m{}^{*}\right)=\frac{9}{20}E_m\left(1-c\right)^{12/7}\ . \tag{46}$$

Case IV (another variant of the case II described in section 2.1 does not represent a limiting case. It is an eligible example which illustrates in complementary way the features of the solutions of the system of equations (14), (15) for composite elastic moduli ν_{eq}, E_{eq} and their connection with bulk and shear moduli as well.

The lower and the upper bounds of elastic moduli for metal matrix composite saturated with spherical voids, corresponding to Case I and Case III are shown on Fig.2.

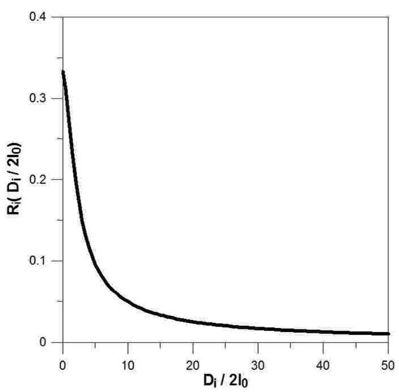

Figure 1. Size sensitivity of function $R_i\left(\eta_i\right)$.

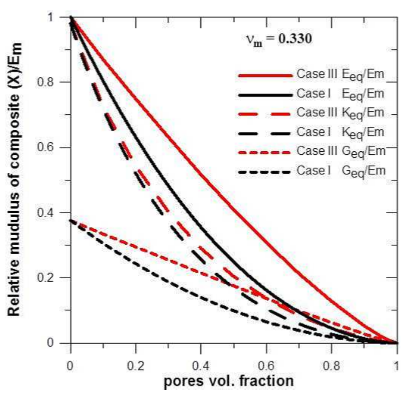

Figure 2. Lower and upper bound of composite moduli accounting for size sensitivity of a metal matrix with $\nu_m = 0.33$. Case I - black curves; Case III - red curves.

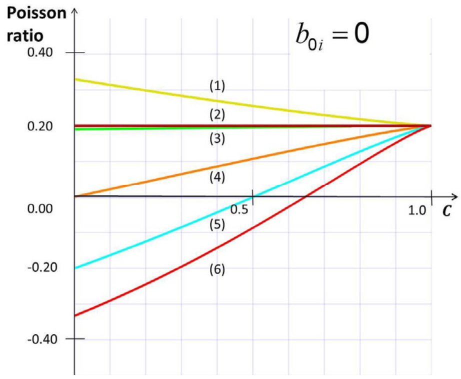

Figure 3. Dependence of Poisson's ratio on inclusions vol. fraction c at different initial matrix Poisson's ratio, Case I; (1) : $\nu_m = 0.33$; (2) : $\nu_m = \nu_m^* = 1/5$; (3) : $\nu_m = 0.19$; (4) : $\nu_m = 0$; (5) : $\nu_m = -0.2$; (6) : $\nu_m = -0.333$.

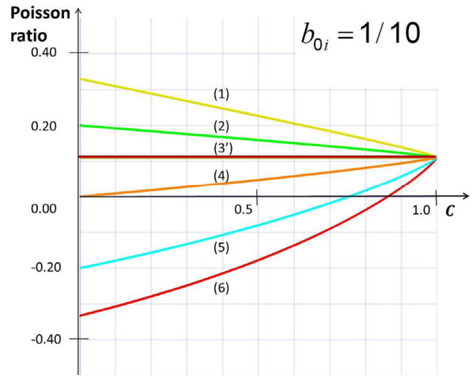

Figure 4. Dependence of Poisson's ratio on inclusions vol. fraction c at different initial matrix Poisson's ratio, Case IV; (1) : $\nu_m = 0.33$; (2) : $\nu_m = 0.2$; (3') : $\nu_m = \nu_m^* = 1/9$; (4) : $\nu_m = 0$; (5) : $\nu_m = -0.2$; (6) : $\nu_m = -0.333$.

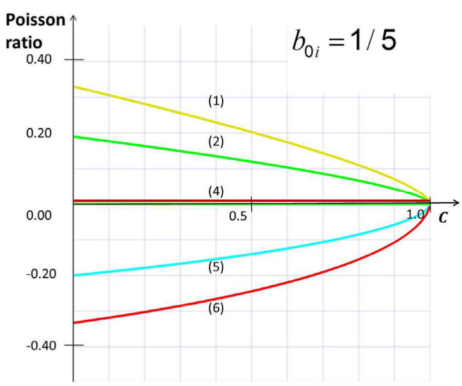

Figure 5. Dependence of Poisson's ratio on inclusions vol. fraction c at different initial matrix Poisson's ratio, Case II; (1) : $\nu_m = 0.33$; (2) : $\nu_m = 0.19$; (4) : $\nu_m = \nu_m^* = 0$; (5) : $\nu_m = -0.2$; (6) : $\nu_m = -0.333$.

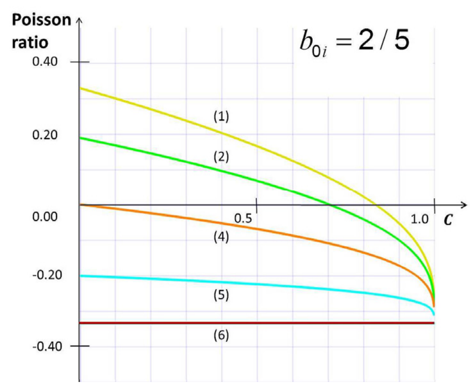

Figure 6. Dependence of Poisson's ratio on inclusions vol. fraction c at different initial matrix Poisson's ratio, Case III; (1) : $\nu_m = 0.33$; (2) : $\nu_m = 0.19$; (4) : $\nu_m = 0$; (5) : $\nu_m = -0.2$; (6) : $\nu_m = \nu_m^* = -1/3$.

As far as the ratios "shear modulus"/"bulk modulus", "Young's modulus"/"bulk modulus" and "Young's modulus"/"shear modulus" are functions of Poisson' ratio exclusively, the solution $\nu_{eq}(c) = \nu_m^* = const$ leads to two remarkable consequences.

1) No matter what amount of pores is added to a matrix with initial elastic moduli (ν_m^*, E_m), where E_m has an arbitrary positive value, the ratio between each pair of moduli of such composite (except for pairs including Poisson's ratio) remains constant for every pore inclusions volume fraction c , $0 \leqslant c \leqslant 1$.

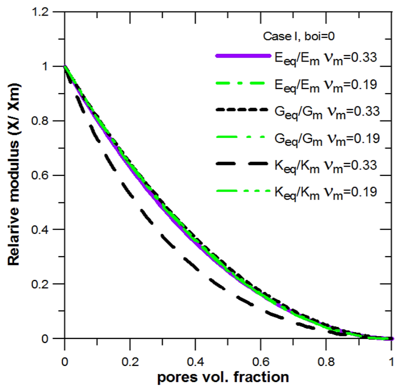

Figure 7. Case I: Dependence of Relative moduli E_{eq}/E_m ; G_{eq}/G_m ; K_{eq}/K_m on inclusions vol. fraction at different initial Poisson's ratio: $\nu_m = 0.33$ - for metal matrix; $\nu_m = 0.19$ - for ceramic matrix.

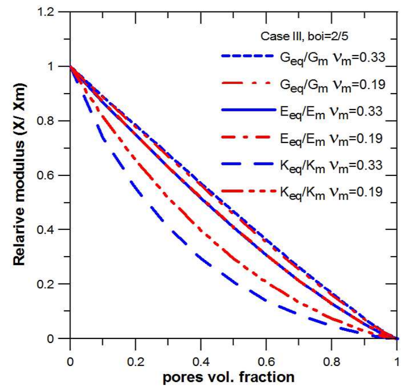

Figure 8. Case III: Dependence of Relative moduli E_{eq}/E_m ; G_{eq}/G_m ; K_{eq}/K_m on inclusions vol. fraction at different initial Poisson's ratio: $\nu_m = 0.33$ - for metal matrix; $\nu_m = 0.19$ - for ceramic matrix.

2) For any admissible b_{0i} there exists a special value of Poisson's ratio ${\nu_m}^* = (1 - 5b_{0i})/5(1 - b_{0i})$ which is connected to the main solution of eqn. (14) for overall Poisson's ratio . For a given b_{0i} and arbitrary chosen matrix moduli (ν_m , E_m) the composite Poisson's ratio tends to ${\nu_m}^*$ at very high pores volume fraction:

$$\lim_{c \to 1} \left[\nu_{eq}(c, b_{0i}) \right] = {\nu_m}^* (b_{0i}) \ . \tag{47}$$

This tendency for the case $b_{oi} = 0$ has been noticed in [8] without any relation to the solution $\nu_{eq}(c) = {\nu_m}^*$ of the eqn. $d\nu_{eq} = 0$, see (20), (21). Earlier in [22] also it has been discussed that at sufficiently high porosity Poisson's ratio converges to a fixed non-zero value, irrespective of the matrix Poisson's ratio. Mentioned limiting behavior of equivalent Poisson's ratio is illustrated on Fig.3,4,5,6 and could be seen directly from the analytical results, presented for cases I-IV by equations (23),(29),(35),(41), respectively. If the initial reactivity of a matrix via the parameter b_{oi} is known, than even without information about its classical elastic moduli, Poisson's ratio bound of the corresponding porous material at very high porosity can be predicted apriory by means of eqn. (19).

The cases I, II, III and IV show up the role of Poisson's ratio in this model. The analytic solutions for $\nu_{eq}(\nu_m, b_{0i}, c\ )$ and corresponding $E_{eq}(E_m, b_{0i}, c\ )$ enable relationships porosity - elastic moduli of all kinds to be dipper investigated and many new aspects of elastic behavior of light composite to be discovered. For example, it is useful to compare two very popular groups of composites - one based on metal matrix, the other - based on ceramics. Despite of observable differences between $\nu_{eq}(\nu_m = 0.3, c\ )$ and $\nu_{eq}(\nu_m = 0.19, c\ )$, (see yellow curves against green curves on Figures 3 - 6) the results obtained indicate that relative Young's moduli and shear moduli have very similar behavior with respect to increasing porosity but relative bulk moduli are highly distinguished for both groups. Such calculations are illustrated on Fig. 7 for the Case I and on Fig. 8 - for the Case III, respectively.

4 Conclusions

A new modification of DEM is presented accounting for inclusion's size effects. The applied approach is based on the size-sensitive dilute homogenization procedure of [16],[17] for two-phase composites.

Analytical solutions ate obtained for elastic properties of a composite containing higher volume fraction of spherical pores, which are suitable for modelling of closed sell foams. Special formulae have been presented which estimate the influence of pore size on elastic behavior of a porous composite by the lower bound (Case I, no size effect) and the upper bound restricting the effective moduli from above, (Case III).

The new variant of DEM described above takes into account inclusions size effects practically in all ranges through the initial internal parameter b_{0i}, see eqn. (13). This model parameter enables materials with equal Cauchy moduli but different microstructure arrangement to be compared as potential matrices of composites with spherical inclusions.

Analytical solutions presented outline some new areas where the desired improvement of mechanical properties of the light composite materials can be achieved and controlled by a proper combination of volume fraction, properties and size parameters of matrix and embedded phases. The model is applicable for metal matrix as well as for non-metal matrix composites and will be developed further for composites with high volume fraction of inclusions of various shapes.

Acknowledgement

The financial support from BG FSI through the grant DH 07/17/ 15 Dec 2016 (project H 07/37/2016) 'New approach for structure and properties design of amorphous and nanostructured metallic foams' is gratefully acknowledged.

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