Graph Theory Towards Module-K Odd-Elegant Labelling Of Graphical Passwords

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Abstract. Graph labellings have been applied in many areas of science and engineering, such as in the development of redundant arrays of independent disks which incorporate redundancy utilizing erasure codes, algorithms, design of highly accurate optical gauging systems for use on automatic drilling machines, design of angular synchronization codes, design of optimal component layouts for certain circuit-board geometries, and determining configurations of simple resistor networks which can be used to supply any of a specified set of resistance values. Based on the idea of “topological structure plus number theory” and magic type of various labellings for solving network transfer delay by using other type of graphical passwords. We define a new graph labelling, called *Module-K odd-elegant labelling*, and find some network models that admit our new labelling, and furthermore our methods can be transformed into effective algorithms.

1 Introduction

No report tells us that much of graphical passwords were applied to business and practice (Ref. [1], [2], [3]). An idea of “topological structure plus number theory” (Topsnut) as an alternative the existing graphical passwords is proposed in [6], which can be realized by graph labellings, and this idea is related with many mathematical conjectures, such as “Every tree is graceful” due to Rosa [4]. The survey article [5] has collected over 1700 papers on research of graph labellings. Motivated from the some graph labellings we define a new labeling, called Module-K odd-elegant labelling. The Module-K odd-elegant labelling can induces some mathematical problems, such as: *Every tree admits a Module-K odd-elegant labelling*. Furthermore, we can plant this new labelling to other graph labellings, see [7], strongly graceful labelling

in [10], odd-graceful labelling in [11], odd-elegant labelling in [18], seven other labellings in [13] and so on, which mean that exploring new graphical passwords will bring more new mathematical subjects and new problems. An idea of “topological structure plus number theory” for creating new type of graphical passwords was proposed firstly by Wang *et al.* in [14] and [15]. In comparing with those graphical passwords mentioned in [16], the new graphical passwords obtained by Topsnut needs less storage and implements quickly in network communication.

2 Preliminary and new labellings

We use standard notation and terminology of graph theory. Graphs mentioned are loopless, no multiple edges, undirected, connected and finite, unless otherwise specified. A (p, q) -graph

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G is one with p vertices and q edges. The shorthand symbol $[m, n]$ stands for an integer set $\{m, m+1, \dots, n\}$, where m and n are integers with $0 \leq m < n$; the notation $[s, t]^o$ indicates an odd-set $\{s, s+2, \dots, t\}$, where s and t both are odd integers with $1 \leq s < t$; and the notation $[k, \ell]^e$ indicates an even-set $\{k, k+2, \dots, \ell\}$, where k and ℓ both are even integers with $0 \leq k < \ell$. Graphical password will be abbreviated as GPW, and “Topsnut-GPW” is the abbreviation of “graphical passwords based on the idea of topological structure plus number theory”.

Definition 1 [5] *If a (p, q) -graph G admits a mapping $f: V(G) \rightarrow [0, q]$ such that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and the edge labelling set $\{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = [1, q]$, then f is called a graceful labelling, and G is called a graceful graph.*

We write the vertex labelling set $f(V(G)) = \{f(u) : u \in V(G)\}$ and the edge labelling set $f(E(G)) = \{f(uv) : uv \in E(G)\}$ hereafter. Suppose that a bipartite graph G admits a graceful labelling f such that $\max\{f(x) : x \in X\} < \min\{f(y) : y \in Y\}$, where (X, Y) is the bipartition of $V(G)$, we call f a set-ordered graceful labelling, and this case is denoted as $f(X) < f(Y)$ for short ([17, 18]).

Definition 2 [9] *Suppose that a (p, q) -graph G admits a mapping $f: V(G) \rightarrow [0, 2q-1]$ such that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and the label $f(uv)$ of every edge $uv \in E(G)$ is defined as $f(uv) = f(u) + f(v) \pmod{2q}$ and the set of all edge labels is equal to $[1, 2q-1]^o$, we call f an odd-elegant labelling and G to be odd-elegant.*

Definition 3 [9] *Let (V_1, V_2) be the bipartition of a bipartite (p, q) -graph G . If G admits an odd-elegant labelling f such that $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$, then we call f a set-ordered odd-elegant labelling, and write this case as $f(V_1) < f(V_2)$.*

Motivated from the above graph labellings we define a new Module-K odd-elegant labelling as follows.

Definition 4 *Suppose that a (p, q) -graph G admits a mapping $f: V(G) \rightarrow [0, 2q-1]$ such that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and the label $f(uv)$ of every edge $uv \in E(G)$ is defined as $f(uv) = f(u) + f(v) + K \pmod{2q}$ for some given K , and the set of all edge labels is equal to $[1, 2q-1]^o$ or $[2, 2q]^e$, we call f a Module-K odd-elegant labelling and G to be Module-K odd-elegant. Moreover, if (V_1, V_2) is the bipartition of G , and the Module-K odd-elegant labelling f obeys $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$, then we call f a set-ordered Module-K odd-elegant labelling, and write this case as $f(V_1) < f(V_2)$.*

In the current paper, we make the following contributions to the literature on password security:

- (1) be used conveniently in usually;
- (2) with strong security, that is, it is difficult to be broken;
- (3) there are enough graphs and labellings for making desired keys and locks.

For answering the above problems, we prove the series cryptographical graphs have good properties, and show the guarantee for constructing large scale of cryptographical trees from smaller cryptographical trees. The methods used for constructing the desired cryptographical graphs can be transformed into efficient algorithms.

3 Main results and their proofs

Theorem 1 *A connected bipartite graph G is set-ordered graceful if and only if there exists a K such that G admits a Module-K odd-elegant labelling.*

Proof. *Necessity.* Suppose that a bipartite graph G admits a graceful labelling f such that $\max\{f(x) : x \in X\} < \min\{f(y) : y \in Y\}$, where (X, Y) , we define $X = \{x_i : i \in [1, s]\}$ and $Y = \{y_j : j \in [1, t]\}$ with $s+t = p$ and $x_i, y_j \in V(G)$. Let G have a set-ordered graceful labelling f with $f(x_i) = i-1$ for $i \in [1, s]$ and $f(y_j) = s-1+j$ for $j \in [1, t]$, and $f(x_i y_j) = f(x_i) - f(y_j) = s+j-i-2$ for each edge

$x_i y_j \in E(T)$. Let G have a labelling g define as: $g(x_i) = 2f(x_i)$ for $i \in [1, s]$, $g(y_j) = 2p - 1 - 2f(y_j)$ for $j \in [1, t]$. for each edge $x_i y_j \in E(G)$, For some given $K \geq 0$, $g(x_i) + g(y_j) + K = 2f(x_i) + 2p - 1 - 2f(y_j) + K = 2(i + p + j - s) + K - 1$, and $g(x_i y_j) = g(x_i) + g(y_j) + K \pmod{2q}$, which implies $g(x_i y_j) \in [1, 2p - 1]^o$ or $[2, 2q]^e$, such that g is Module-K odd-elegant labelling with for each edge $x_i y_j \in E(G)$.

Sufficiency. Suppose that G admits a Module-K odd-elegant labelling h such that $h(x_i) = 2(i - 1)$, $i \in [1, s]$; $h(y_j) = 2p - 1 - 2(s - 1 + j)$, $j \in [1, t]$. $h(x_i y_j) = 2(i + p - s - j) - 1 + K \pmod{2q}$ for $i \in [1, s]$ and $j \in [1, t]$, and $h(x_i y_j) \in [1, 2q - 1]^o$ or $[2, 2q]^e$. We define a labelling f_1 of G as: $f_1(x_i) = \frac{1}{2}h(x_i)$ for $i \in [1, s]$; $f_1(y_j) = h(y_j) - 2p + 1$ for $j \in [1, t]$; $f_1(x_i y_j) = h(x_i y_j) - K$ for $i \in [1, s]$ and $j \in [1, t]$. which produces $f_1(V(G)) = [1, q]$ and $f_1(x_i y_j) \in [0, q]$. Thereby, f_1 is a set-ordered graceful labelling of G .

Figure 1. (a) A Module-8 odd-elegant labelling; (b) a Module-6 odd-elegant labelling.

A union graphical quasi-group $\cup_i(G)$ of order n is defined as: $\cup_i(G) = \{G : V(G) = 1, 2, \dots, n\}$, the operation between elements of $\cup_i(G)$ is the graph union operation “ $\cup$ ”, and $G \cup H$ produces a simple graph of order n .

(i) (Zero element) the zero element of $\cup_i(G)$ is the unique complete graph K_n , since $G \cup K_n = K_n$.

(ii) (Inverse element) the complementary G of each element $G \in \cup_i(G)$ is the inverse element of G , since $G \cup G = K_n$.

(iii) Uniqueness and closure. $G \cup H$ is unique and $G \cup H \in \cup_i(G)$ for any element $H \in \cup_i(G)$.

(iv) (Associated law) $(G_1 \cup G_2) \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

A graphical group obtained from the graph Module-K odd-elegant labelling. Let $f_1 :$

$V(G_k) \rightarrow [0, 2q - 1]$ be a Module-K odd-elegant labelling such that the set of all edge labels is equal to $[1, 2q - 1]^o$ or $[2, 2q]^e$ for a (p, q) -graph G_k . Define every edge $uv \in E(G_k)$ is defined as $f(uv) = f(u) + f(v) + K \pmod{2q}$ with some given K . Notice that $T_i (i \in [1, n])$ has identical vertex labelling of G_k . Each copy of the (p, q) -graph G_k is written as $T_i (i \in [1, n])$. Define a 3-element operation $T_i \oplus T_j \ominus T_k = T_{i+j-k \pmod{K}}$. We call this particular operation as the Module-K odd-elegant additive-sum of graphs.

Figure 2. $T_i (i \in [1, 4])$.

Corollary 1 The set $Oele(G)$ based on a (p, q) -graph G_1 forms an Abelian additive group, we call $Oele(G)$ a labelling graphical group based on the graph G hereafter. Furthermore, $f(uv) = f(u) + f(v) + K \pmod{2q}$ with some given K .

Proof. First of all, the equation $F(T_i) \oplus F(T_j) \ominus F(T_k) = F(T_{i+j-k \pmod{2q}})$ with $f(uv) \pmod{2q} \in [1, 2q - 1]^o$ or $[2, 2q]^e$ $f(uv) = f(u) + f(v) + K \pmod{2q}$ follows the equation $[f_i(x) + f_j(x) + K] \pmod{2q}$ with $V(G) = V(G_i) = V(G_j) = V(G_k)$. We are ready to show four principles for a standard group in the following.

(1) Zero element. The zero element of $Oele(G)$ is $F(T_1)$ as $k = 0$ in the equation.

(2) Inverse element. The inverse of each element $T_i \in Oele(G)$. In fact, $F(T_i) \oplus F(T_j) \ominus F(T_k) = F(T_{i+j-k \pmod{2q}})$.

(3) Uniqueness and closure. $F(T_i) \oplus F(T_j) \in Oele(G)$.

(4) Associative law. Since $[F(T_i) \oplus F(T_j)] \oplus F(T_l) = F(T_{i+j-l \pmod{2q}})$, so the associative law holds true.

We conclude that $Oele(G)$ is a Module-K odd-elegant graphical group. Next, we determine a connection between $f_1(uv) = f_1(u) +$

$f_1(v) + K(\bmod 2q)$ and $f_k(u) + f_k(v) + K(\bmod 2q)$ with some given K in the following. Notice that $f_1(E(T_1)) = [1, 2q-1]^o$ or $[2, 2q]^e$. Consider the labelled graph T_k , take an edge $uv \in E(T_k)$, we have $f_k(uv) = f_k(u) + f_k(v) + K(\bmod 2q) = [f_1(u) + f_1(v) + (k-1)](\bmod 2q)$, and there are the following cases: $f_k(u) + f_k(v) = f_1(u) + f_1(v)$. The proof of this theorem is complete.

Theorem 2 Let $G_1, G_2, \dots, G_m$ be m bipartite graphs having Module-K odd-elegant labellings and (X_i, Y_i) be the bipartition of each $V(G_i)$ with $i \in [1, m]$. The graph G is obtained by joining a vertex of G_i with a vertex of G_{i+1} for $i \in [1, m]$. When m is odd, Then G has a Module-K odd-elegant labelling.

Figure 3. A Module-2 odd-elegant labelling for illustrating Theorem 2.

A dual labelling f_k of a modular labelling f_k is defined by $f_k(uv) \in [1, 2q-1]^o$ or $[2, 2q]^e$. Let $Oele(G)$ be the set of the graphs H_k , where each H_k is a labelled $Oele(G)$ by the dual labelling f_k of the modular labelling f_k with $x \in V(G)$ with some given K . We propose two new graphical subject to some con-straint sets as: When $G_1, G_2, \dots, G_m$ be m bipartite graphs and neither m is odd or even with the Module-K odd-elegant labelling g induced by the vertex set-labelling, we show that every simple, connected

(p, q) -graph G admits a Module-K odd-elegant labelling, and any tree having a Module-K odd-elegant labelling. We propose the following problems for further researching new graphical passwords:

Theorem 3 Let $G_1, G_2, \dots, G_m$ be m bipartite graphs having Module-K odd-elegant labellings and (X_i, Y_i) be the bipartition of each $V(G_i)$ with $i \in [1, m]$. The graph G is obtained by joining a vertex of G_i with a vertex of G_{i+1} for $i \in [1, m]$. Then G has a Module-K odd-elegant labelling.

Theorem 4 Let $G_1, G_2, \dots, G_m$ be m bipartite graphs having Module-K odd-elegant labellings and (X_i, Y_i) be the bipartition of each $V(G_i)$ with $i \in [1, m]$. The graph G is obtained by joining a vertex of G_i with a vertex of G_{i+1} for $i \in [1, m]$. If m is even, then G has a Module-K odd-elegant labelling.

4 Conclusion and further researches

We are working on new type of graphical passwords based on the idea of “topological structure plus number theory”, vividly speaking, new graphical passwords can be called “mathematical fingerprints”. And we are exploring transforming human faces into mathematical models in order to recognizing and comparing human faces through mathematical models rather than image recognition among vast of human faces, such works can be considered as “mathematical faces”. However, we will make effort to let mathematical fingerprints approximating to human fingerprints. By our experience on working Topsnut-GPWs, we propose the following conjectures:

- (i) Every bipartite graph admits a Module-K odd-elegant labelling.
- (ii) A graph is odd-graceful if and only if it admits a Module-K odd-elegant labelling.
- (iii) Any tree T has a Module-K odd-elegant labelling.

Figure 4. A Module-60 odd-elegant labelling for illustrating Theorem 4

Some of recent works have demonstrated that textual passwords are particularly vulnerable to targeted online guessing. Textual passwords may be influenced by personal information and textual passwords may be reused, both of which contribute to the vulnerability of textual passwords to online guessing. “Topological structure plus number theory” can be realized by graph labellings, future tests could be designed to answer this question.

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