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On the problem of shear of a functionally graded half-space by a punch

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Abstract. Contact problem on shear of a functionally graded half-space by a strip bunch was considered. Shear modulus of the half-space is exponentially increasing by depth. The contact problem was reduced to a convolution integral equation of the first kind. Solution of the integral equation was constructed by asymptotic methods over the characteristic parameter of the problem. Dependence of the obtained problem solution and its main characteristics on the shear modulus of the half-space was analysed. Expressions for the main characteristics of the problem are given.

1 Introduction

From the mid-20th century functionally graded inhomogeneity of materials is accounted for in analysis of plates, foundations [1, 2] etc. In modern times, FGMs, apart from the civil engineering, are widely used in industry, especially in microelectronics [3, 4], where traditional materials like alumina, silicon etc. are oxidized to form an insulator layer. Oxides can have elastic modulus two times or more as the bulk material, their thickness can present a sufficiently large amount of the component thickness. Therefore, analysis of stress and strain, electrostatic or magnetostatic state of such materials should take into account variation of elastic moduli, electric and magnetic parameters in volume of the material. Common analytical approach consist in solution of elasticity problems using reduction to an integral equation. Various methods were developed for solution of these equations, including orthogonal polynomials method [5], collocation method [6], bilateral asymptotic method [7]. These approaches are widely used nowadays to get the solution of the contact problems for FGMs [8–15]. However, improvement of solutions' accuracy needs development of new analytical approaches to FGM mechanics modelling. In the present work, an elastic contact problem is considered on a pure shear of a functionally graded half-space by a punch. Solution of the problem is reduced to the integral equation. Approximated analytic solution is constructed by asymptotic methods. Dependence of main mechanical characteristics of the problem on the half-space shear modulus parameters is analysed.

2 Problem statement

The rigid strip punch with width $2a$ ($|x| \le a$, $y = 0$) shears by the amount ε ($w(x,0) = \varepsilon$, $|x| \le a$) the surface of the elastic half-space made of functionally graded material with shear modulus dependent on depth $\mu(y) = \mu e^{2dy}$ ($0 < y < \infty$), where $\mu(0) = \mu$ is shear modulus on the half-space surface ($y = 0$), d is the gradient parameter ($d > 0$). Contact stresses $\varphi(x) = -\sigma_{yz}(x,0)$ arising under the punch are to be determined, along with displacements $w(x,0)$ of the half-space surface outside of the contact region ($a < |x| < \infty$). It is assumed that displacements of the half-space $w(x, y)$ decay at infinity.

In the half-space, the pure shear deformation conditions are fulfilled: $\sigma_{xx} = \sigma_{xy} = \sigma_{yx} = \sigma_{yy} = \sigma_{zz} = 0$, where σ_{xx}, σ_{yy}, σ_{zz} are normal stresses, σ_{xy}, σ_{yx} are tangential stresses [16].

In the considered case, elasticity equations are simplified to the equation

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \qquad (1)$$

where σ_{zx}, σ_{zy} are tangential stresses, which can be expressed in terms of strains

$$\sigma_{zx} = \mu(y)\frac{\partial w}{\partial x}, \quad \sigma_{zy} = \mu(y)\frac{\partial w}{\partial y} \qquad (2)$$

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where $w(x,y)$ are displacements of the half-space along the z axis.

Substituting (2) to the equation (1), differential equation of elasticity in terms of displacements is obtained:

$$\frac{\partial^2 w}{\partial y^2}+\frac{\mu'(y)}{\mu(y)}\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial x^2}=0 \qquad (3)$$

Taking into account that $\mu(y)=\mu e^{2dy}$, the differential equation (3) with variable coefficient dependent on y turns into the differential equation with constant coefficients

$$\frac{\partial^2 w}{\partial y^2}+2d\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial x^2}=0 \qquad (4)$$

Mixed boundary conditions of the problem have the form

$$y=0 \qquad w(x,0)=\varepsilon \qquad |x|\le a \qquad (5)$$

$$\sigma_{yz}(x,0)=\begin{cases}-\varphi(x) & |x|\le a\\ 0 & a<|x|<\infty\end{cases} \qquad (6)$$

where $\varphi(x)$ are unknown contact stresses under the punch.

At infinity displacements $w(x,y)$ turns to zero, i.e. $\lim\limits_{\sqrt{x^2+y^2}\to\infty} w(x,y)=0$.

3 Integral equation of the problem

Using the Fourier integral transform

$$w^F(\alpha,y)=\int_{-\infty}^{\infty} w(x,y)e^{i\alpha x}\mathrm{d}x\,, \qquad w(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} w^F(\alpha,y)e^{-i\alpha x}\mathrm{d}\alpha \qquad (7)$$

where F index above w stands for the Fourier transform of the $w(x,y)$ function, the differential equation (4) is reduced to an ordinary differential equation with constant coefficients

$$\frac{\mathrm{d}^2 w^F}{\mathrm{d}y^2}+2d\frac{\mathrm{d}w^F}{\mathrm{d}y}-\alpha^2 w^F(\alpha,y)=0 \qquad (8)$$

The general solution of this equation decaying at infinity ($y\to\infty$) has the form

$$w^F(\alpha,y)=C(\alpha)e^{\lambda_- y}\,,\ \lambda_-=-d-\sqrt{d^2+\alpha^2}\ ,\ d>0 \quad (9)$$

where $C(\alpha)$ is the unknown constant determined from the boundary condition (6) after applying the Fourier transformation (7) to it:

$$C(\alpha)=-\mu^{-1}(0)\varphi^F(\alpha)\exp(-\lambda_- y)/\lambda_-$$

where $\varphi^F(\alpha)$ is the image of $\varphi(x)$.

Determining $w(x,y)$ using the inverse Fourier transform in (9) and satisfying mixed boundary conditions (5), (6), the integral equation of the problem was obtained

$$\int_{-a}^{a}\varphi(\xi)k(\xi-x)\mathrm{d}\xi=2\pi\mu(0)\varepsilon \qquad |x|\le a \qquad (10)$$

$$k(x)=\int_{-\infty}^{\infty}K(\alpha)e^{i\alpha t}\mathrm{d}\alpha\ ,\ K(\alpha)=\left(d+\sqrt{d^2+\alpha^2}\right)^{-1} \quad (11)$$

The integral equation (10), (11) was then represented in dimensionless formulation

$$\int_{-1}^{1}\varphi(\xi)k\left(\frac{\xi-x}{\lambda}\right)\mathrm{d}\xi=2\pi\frac{\mu(0)}{a}\varepsilon \quad |x|\le 1,\ \lambda=\frac{1}{da} \quad (12)$$

$$k(t)=\int_{-\infty}^{\infty}K(\alpha)e^{i\alpha t}\mathrm{d}\alpha\ ,\ \ K(\alpha)=\left(1+\sqrt{1+\alpha^2}\right)^{-1} \quad (13)$$

being the Fourier convolution integral equation of the fist kind with differential kernel. The $K(\alpha)$ function is the Fourier image of the integral equation kernel $k(t)$ and is even, multivalued in the complex plane with algebraic branch points $\alpha=\pm i$. $K(\alpha)$ express the following asymptotic properties

$$\begin{aligned}K(\alpha)&=|\alpha|^{-1}+\mathrm{O}(\alpha^{-2}) \quad \text{at } |\alpha|\to\infty\\ K(\alpha)&=K(0)+\mathrm{O}(\alpha^2) \quad \text{at } |\alpha|\to 0\end{aligned} \qquad (14)$$

where $K(0)=1/2$. Calculation of (13) gives the following representation for the kernel $k(t)$

$$k(t)=2\left(K_0(t)-\int_t^{\infty}\mathrm{d}\tau\int_{\tau}^{\infty}K_0(\xi)\mathrm{d}\xi\right)e^t \qquad (15)$$

where $K_0(t)$ is the Macdonald function. At small t the following estimation takes place

$$k(t)=-\ln|t|+\mathrm{O}(1) \qquad \text{at } t\to 0 \qquad (16)$$

At the fulfillment of the above stated properties of the kernel $k(t)$ the integral equation (12) has the unique solution [17].

4 Asymptotic solution of the integral equation

The most effective analytic methods of the integral equation (12) solution are asymptotic methods with respect to the dimensionless parameter $\lambda = da$ for its small ($0 < \lambda < 1$) or large values ($\lambda > 2$) [17].

At small values of λ the solution of the integral equation (12) is constructed in the form of the Neumann series with zeroth term represented as

$$\varphi(x) = \varphi_+\left((1+x)\lambda^{-1}\right) + \varphi_-\left((1-x)\lambda^{-1}\right) - \varphi_\infty(x\lambda^{-1}) \quad (17)$$

where functions $\varphi_\pm(x)$ are determined from the Wineher-Hopf integral equation on the half-axis [18]

$$\int_0^\infty \varphi_\pm(\xi)k(\xi - x)\mathrm{d}\xi = 2\pi \frac{\mu(0)}{a\lambda}\varepsilon \quad 0 < x < \infty \quad (18)$$

while function $\varphi_\infty(x)$ is determined from the convolution integral equation on the infinite interval

$$\int_{-\infty}^\infty \varphi_\infty(\xi)k(\xi - x)\mathrm{d}\xi = 2\pi \frac{\mu(0)}{a\lambda}\varepsilon \quad -\infty < x < \infty \quad (19)$$

Solution of the integral equation (19) is constructed using the Fourier integral transform (7) and has the form

$$\varphi_\infty(x) = \frac{\mu(0)}{a\lambda}\frac{\varepsilon}{K(0)} \quad -\infty < x < \infty \quad (20)$$

Solution of the integral equations (18) on the half-axis is constructed by the Wiener-Hopf method. On the first stage, equation (18) is extended to the whole axis *x*

$$\int_0^\infty \varphi_\pm(\xi)k(\xi - x)\mathrm{d}\xi = \begin{cases} 2\pi \dfrac{\mu(0)}{a\lambda}\varepsilon & 0 < x < \infty \\ 2\pi \dfrac{\mu(0)}{a\lambda} w_\mp(x) & -\infty < x < 0 \end{cases} \quad (21)$$

where $w_\mp(x)$ are given by operators

$$w_\mp(x) = \int_{-\infty}^0 \varphi_\pm(\xi)k(\xi - x)\mathrm{d}\xi \quad -\infty < x < 0$$

which present displacements of the half-space free surface outside the contact region.

On the first stage of the Wiener-Hopf method, the Fourier transform (7) is applied to solve integral equation on the whole axis (21), performing transition of the latter to functional equations in transforms $\varphi_\pm^F(\alpha)$ and $w_\mp^F(\alpha)$

$$\varphi_\pm^F(\alpha)K(\alpha) = \frac{\mu(0)}{\lambda a}\varepsilon\left(-\frac{1}{ia}\right) + \frac{\mu(0)}{\lambda a} w_\mp^F(\alpha)$$
$$\eta_- < \mathrm{Im}(\alpha) < \eta_+ \quad (22)$$

where

$$\varphi_\pm^F(\alpha) = \int_0^\infty \varphi(\xi)e^{i\alpha\xi}\mathrm{d}\xi \ , \quad w_\mp^F(\alpha) = \int_{-\infty}^0 w_\mp(\xi)e^{i\alpha\xi}\mathrm{d}\xi$$

are unknown transforms to be determined from (22). Assuming that factorization of $K(\alpha)$ is known, i.e.

$$K(\alpha) = K_+(\alpha)K_-(\alpha) \quad (23)$$

where $K_+(\alpha)$ is regular at $\mathrm{Im}(\alpha) > \eta_-$, $K_-(\alpha)$ is regular in half-plane $\mathrm{Im}(\alpha) < \eta_+$ ($\eta_+ > \eta_-$). Substituting (23) to (22), left- and right-hand sides of the obtained equation was divided by $K_-(\alpha)$. The function $f(\alpha) = \left(i\alpha K_-(\alpha)\right)^{-1}$ was then represented in the form of composition

$$f(\alpha) = \frac{1}{i\alpha K_-(\alpha)} = f_+(\alpha) + f_-(\alpha)$$

where $f_+(\alpha)$ is regular in the upper half-plane $\mathrm{Im}(\alpha) > \eta_-$, while $f_-(\alpha)$ is regular in the lower half-pane $\mathrm{Im}(\alpha) < \eta_+$, that in in the considered case was fulfilled by elementary means

$$f_+(\alpha) = \frac{1}{K_-(0)i\alpha}\ , \quad f_-(\alpha) = \frac{K_-(0) - K_-(\alpha)}{i\alpha K_-(0)K_-(\alpha)}$$

After this, the functional equation is represented in the form of equality of two functions in the complex plane

$$\varphi_\pm^F(\alpha)K_+(\alpha) + \frac{\mu(0)}{a\lambda}f_+(\alpha) =$$
$$\frac{\mu(0)}{a\lambda}\varepsilon f_-(\alpha) + \frac{\mu(0)}{a\lambda}\frac{w_\mp^F(\alpha)}{K_-(\alpha)} = F(\alpha) \quad (24)$$
$$\eta_- < \mathrm{Im}(\alpha) < \eta_+$$

where left-hand side of (24) is analytical extension to the upper half-plane ($\mathrm{Im}(\alpha) > \eta_-$), and right-hand side to the lower ($\mathrm{Im}(\alpha) < \eta_+$) half-plane of a some $F(\alpha)$. As analytical extensions of $F(\alpha)$ to the upper and lower half-planes decay at infinity $|\alpha| \to \infty$ it follows that $F(\alpha) = 0$ in the complex plane. From (24) two equations follow, from which it was determined that

$$\varphi_\pm^F(\alpha) = -\frac{\mu(0)}{a\lambda}\frac{\varepsilon}{K_-(0)}\frac{1}{i\alpha K_+(\alpha)}$$
$$w_\mp^F(\alpha) = \varepsilon\left(\frac{1}{i\alpha} - \frac{1}{K_-(0)}\frac{K_-(\alpha)}{i\alpha}\right) \quad (25)$$

Inverting the Fourier transform (7) of equations (25), solution of the integral equations (21) were obtained

$$\varphi_{\pm}(x) = -\frac{\mu(0)}{a\lambda}\frac{\varepsilon}{K_-(0)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha x}}{i\alpha K_+(\alpha)}\mathrm{d}\alpha \qquad x>0 \quad (26)$$

$$w_{\mp}(x) = \frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{i\alpha}-\frac{1}{K_-(0)}\frac{K_-(\alpha)}{i\alpha}\right)e^{-i\alpha x}\mathrm{d}\alpha \quad x<0 \quad (27)$$

To calculate integrals in (26), (27) the factorization of the function $K(\alpha)$ from (13), i.e. representation in the form (23), is needed. To obtain compact analytic solution factorization of (23) should be performed by elementary means, which is no case for the function $K(\alpha)$ (13). So the function $K(\alpha)$ is replaced by its approximation

$$K(\alpha) = \left(\sqrt{(K^{-1}(0))^2+\alpha^2}\right)^{-1} \qquad (28)$$

which is close to the exact function on the real axis (which is the axis of integration in (12)) and accomplishes to asymptotic properties (14), (15). Error of such approximation along the real axis do not exceed 13%. Factorization (23) of the approximation (28) is achieved by elementary means in the form

$$K_{\pm}(\alpha) = \left(K^{-1}(0) \mp i\alpha\right)^{-1/2} \qquad (29)$$

where $K_+(\alpha)$ is regular in the upper half-plane $\mathrm{Im}(\alpha)>\eta_-$, $K_-(\alpha)$ is regular in the lower half-plane $\mathrm{Im}(\alpha)<\eta_+$ of the complex plane $\alpha=\xi+i\eta$, where $\eta_{\pm}=\pm K^{-1}(0)$. The function $K_+(\alpha)$ is analytical extension of $K(\alpha)$ from the strip $|\mathrm{Im}(\alpha)|<K^{-1}(0)$ to the upper half-plane, $K_-(\alpha)$ to the lower half-plane. Substituting $K_+(\alpha)$ from (29) and calculating integral in (26) it was obtained that

$$\varphi_{\pm}(x) = \frac{\mu(0)}{a\lambda}\frac{\varepsilon}{K_-(0)}\left(\frac{1}{\sqrt{\pi x}}\exp\left(-K^{-1}(0)x\right)+\sqrt{K^{-1}(0)}\,\mathrm{erf}\left(\sqrt{K^{-1}(0)x}\right)\right) \qquad x>0 \quad (30)$$

Substituting $K_-(\alpha)$ from (29) and calculating integral in (27) it was obtained that

$$w_{\mp}(x) = \varepsilon\,\mathrm{erfc}\left(\sqrt{K^{-1}(0)(-x)}\right) \qquad x<0 \quad (31)$$

where $\mathrm{erf}(x)$ is the error function [19], $\mathrm{erfc}(x)=1-\mathrm{erf}(x)$.

Substituting $\varphi_{\pm}(x)$ from (30) and $\varphi_{\infty}(x)$ from (20) to (17), solution of the integral equation (12) was obtained, together with the problem solution. Functions $w_{\mp}(1\pm x/\lambda)$ in (31) give displacements of stress-free surface of the half-space outside of the contact region to the left

$$w_-\left(\frac{1+x}{\lambda}\right) = \varepsilon\,\mathrm{erfc}\left(2\left(-\frac{1+x}{\lambda}\right)\right) \quad \text{at } x<-1$$

and to the right

$$w_+\left(\frac{1-x}{\lambda}\right) = \varepsilon\,\mathrm{erfc}\left(\sqrt{2\left(-\frac{1-x}{\lambda}\right)}\right) \quad \text{at } x>1$$

from the punch.

Punch shear force T, which is an integral characteristic of the problem, can be obtained by integration of (17) over interval [−1, 1]

$$T = T_+ + T_- - T_\infty,$$
$$T_{\pm} = \int_{-1}^{1}\varphi_{\pm}\left(\frac{1\pm x}{\lambda}\right)\mathrm{d}x,\; T_\infty = \int_{-1}^{1}\varphi_\infty\left(\frac{x}{\lambda}\right)\mathrm{d}x \qquad (32)$$

Taking solution of the integral equation (12) in multiplicative form [17]

$$\varphi(x) = \varphi_+\left(\frac{1+x}{\lambda}\right)\varphi_-\left(\frac{1-x}{\lambda}\right)\Big/\varphi_\infty\left(\frac{x}{\lambda}\right) \qquad (33)$$

and integrating it over interval [−1, 1], one can obtain the formula, convenient for analysis, which is asymptotically equivalent to (32):

$$T = 2\mu(0)\varepsilon(1+\gamma), \quad \gamma = 2/\lambda = 2da \qquad (34)$$

The obtained formulas allow to deduct other important problem's characteristics, such as displacements $w(x,y)$ and stresses $\sigma_{yz}(x,y)$ in the half-space and point out their dependence on shear modulus $\mu(y)$

$$w(x,y) = -\frac{a}{\mu(0)}\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)\theta\left(\frac{\xi-x}{\lambda},dy\right)\mathrm{d}\xi \quad y>0 \quad (35)$$

$$\sigma_{yz}(x,y) = -\frac{da}{\pi}\sqrt{\frac{\mu(y)}{\mu(0)}}\ln\sqrt{\frac{\mu(y)}{\mu(0)}}\times\int_{-1}^{1}\varphi(\xi)\kappa\left(\frac{\xi-x}{\lambda},dy\right)\mathrm{d}\xi \quad y>0 \quad (36)$$

where

$$\theta(u,v) = \int_0^v\sqrt{\frac{\mu(0)}{\mu(\eta/d)}}\ln\sqrt{\frac{\mu(0)}{\mu(\eta/d)}}\kappa(u,\eta)\mathrm{d}\eta$$

$$\kappa(u,v) = \left(u^2+v^2\right)^{-1/2}K_1\left(\sqrt{u^2+v^2}\right)$$

and $K_1(x)$ is a cylinder function [19].

From (36) it follows that shear stresses $\sigma_{yz}(x, y)$ increase, whilst displacements $w(x, y)$ decrease with depth coordinate *y* proportionally to $\sqrt{\mu(y)}$.

To construct solution of the integral equation (12) of the problem at large values of λ (λ > 2) it is sufficient to use results described in [20].

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