Finite difference approximation of electron balance problem in the stationary high-frequency induction discharges

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Abstract. The problem of finding the minimal eigenvalue corresponding to a positive eigenfunction of the nonlinear eigenvalue problem for the ordinary differential equation with coefficients depending on a spectral parameter is investigated. This problem arises in modeling the plasma of radio-frequency discharge at reduced pressures. The original differential eigenvalue problem is approximated by the finite difference method on a uniform grid. A sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of the finite difference nonlinear eigenvalue problem is established. Error estimates for the approximate eigenvalue and the corresponding approximate positive eigenfunction are proved. Investigations of this paper generalize well known results for eigenvalue problems with linear dependence on the spectral parameter.

1 Introduction

In the present paper, we study the nonlinear eigenvalue problem of finding the minimal eigenvalue $\lambda \in \Lambda$, $\Lambda = [0, \infty)$, corresponding to a positive eigenfunction $u(x)$, $x \in \Omega$, $\Omega = (0, \pi)$, $\bar{\Omega} = [0, \pi]$, satisfying the following equations

$$-(p(\lambda s(x))u'(x))' = r(\lambda s(x))u(x), \quad x \in \Omega, \quad (1)$$

$$u(0) = u(\pi) = 0. \quad (2)$$

We assume that $p(\mu)$, $r(\mu)$, $\mu \in \Lambda$, and $s(x)$, $x \in \bar{\Omega}$, are smooth positive functions. We also assume that the function $p(\mu)$, $\mu \in \Lambda$, is nondecreasing and bounded and the function $r(\mu)$, $\mu \in \Lambda$, is nondecreasing and unbounded. The nonlinear eigenvalue problem (1), (2) is approximated by the finite difference method on a uniform grid. A sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of the finite difference nonlinear eigenvalue problem is established. Error estimates for the approximate eigenvalue and the corresponding approximate positive eigenfunction are proved.

Nonlinear eigenvalue problems of the form (1), (2) arise in modeling the plasma of radio-frequency discharge at reduced pressures. An inductive coupled radio-frequency discharge has found broad applications in diverse technological plasma processes, such as processing textiles and leather-fur half-finished products, metals, hydrogen accumulation by silicon powders, synthesis of oxygen-free ceramic materials, and obtaining carbide and boride materials for nuclear and processing industry [1]. A more effective and qualitative

choice of constructive solutions in designing inductive coupled radio-frequency devices requires mathematical models, because some technological characteristics of the plasma cannot be measured.

Nonlinear eigenvalue problems also arise in various fields of science and technology [2-4]. Numerical methods for solving matrix eigenvalue problems with nonlinear dependence on the parameter were constructed and investigated in the papers [5-13]. Mesh methods for solving differential nonlinear eigenvalue problems were studied in [14-16]. The theoretical basis for the study of nonlinear eigenvalue problems is results obtained for linear eigenvalues problems [17-23]. In the papers [24-30], numerical methods for solving applied nonlinear boundary value problems and variational inequalities have been studied.

2 Variational statement of the problem

Let $H = L_2(\Omega)$ denote the real Lebesgue space with norm $|\cdot|_0$ and inner product $(\cdot, \cdot)_0$ defined by

$$|v|_0 = \left(\int_0^\pi (v(x))^2 dx \right)^{1/2}, \quad (3)$$

$$(u, v)_0 = \int_0^\pi u(x)v(x)dx \quad \forall u, v \in H. \quad (4)$$

By $V = \{v : v, v' \in H, v(0) = v(\pi) = 0\}$ we denote the real Sobolev space with norm $|\cdot|_1$ and inner product $(\cdot, \cdot)_1$ defined by

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$$\|v\|_1 = \left(\int_0^\pi (v'(x))^2 dx \right)^{1/2}, \quad (5)$$

$$(u, v)_1 = \int_0^\pi u'(x)v'(x)dx \quad \forall u, v \in V. \quad (6)$$

For $\mu \in \Lambda$, we introduce the bilinear forms defined by the formulas

$$a(\mu, u, v) = \int_0^\pi p(\mu s(x))u'v'dx, \quad (7)$$

$$b(\mu, u, v) = \int_0^\pi r(\mu s(x))uv'dx, \quad (8)$$

for any $u, v \in V$, and the Rayleigh ratio

$$R(\mu, v) = \frac{a(\mu, v, v)}{b(\mu, v, v)} \quad \forall v \in V \setminus \{0\}. \quad (9)$$

Set $K = \{v : v \in V, v(x) > 0, x \in \Omega\}$.

The differential eigenvalue problem (1), (2) is equivalent to the following variational eigenvalue problem: find the least $\lambda \in \Lambda$, $u \in K$, $b(\lambda, u, u) = 1$, such that

$$a(\lambda, u, v) = b(\lambda, u, v) \quad \forall v \in V. \quad (10)$$

For fixed $\mu \in \Lambda$, we introduce the following linear parametric variational eigenvalue problem: find the least $\gamma(\mu) \in \mathbb{R}$, $u \in K$, $b(\mu, u, u) = 1$, such that

$$a(\mu, u, v) = \gamma(\mu)b(\mu, u, v) \quad \forall v \in V. \quad (11)$$

The following variational property is valid

$$\gamma(\mu) = \min_{v \in V \setminus \{0\}} R(\mu, v). \quad (12)$$

Let $\gamma_i(\mu)$, $u_i(\mu) = u_i(\mu, x)$, $x \in \bar{\Omega}$, $\mu \in \Lambda$, $i = 1, 2, \dots$, be eigenvalues and eigenfunctions satisfying equation (3),

$$\gamma_1(\mu) < \gamma_2(\mu) < \dots < \gamma_i(\mu) < \dots, \quad (13)$$

$$a(\mu, u_i(\mu), u_j(\mu)) = \gamma_i(\mu)\delta_{ij}, \quad (14)$$

$$b(\mu, u_i(\mu), u_j(\mu)) = \delta_{ij}, \quad (15)$$

$i, j = 1, 2, \dots$, $u_i(\mu) \in K$, $\gamma_1(\mu) = \gamma(\mu)$, $\gamma_i(\mu) \rightarrow \infty$ as $i \rightarrow \infty$, functions $u_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \dots$, form a complete system in the space V . The minimal eigenvalue λ of problem (10) is the least root of the equation $\gamma(\mu) = 1$, $\mu \in \Lambda$.

Theorem 1. If $p(0) > r(0)$, then there exists a minimal simple eigenvalue of problem (10) corresponding to a positive eigenfunction.

The result of Theorem 1 is proved by analogy with [15, 16].

3 Finite difference approximation of the problem

Let us partition the interval $[0, \pi]$ by equidistant points $x_i = ih$, $i = 0, 1, \dots, N$, into elements $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \dots, N$, $h = \pi/N$, and let V_h denote the subspace of the space V consisting of continuous functions v^h linear on each element e_i , $i = 1, 2, \dots, N$. For $\mu \in \Lambda$, $u^h, v^h \in V_h$, we introduce approximate bilinear forms

$$a_h(\mu, u^h, v^h) = \sum_{i=1}^N hp(\mu s(x_i - h/2))(u^h(x_i - h/2))'(v^h(x_i - h/2))', \quad (16)$$

$$b_h(\mu, u^h, v^h) = \sum_{i=1}^N h(r(\mu s(x_{i-1}))u^h(x_{i-1})v^h(x_{i-1}) + r(\mu s(x_i))u^h(x_i)v^h(x_i))/2. \quad (17)$$

Set $K_h = \{v^h : v^h \in V_h, v^h(x) > 0, x \in \Omega\}$.

We approximate the original eigenvalue problem (10) by a finite-dimensional eigenvalue problem: find the least $\lambda^h \in \Lambda$, $u^h \in K_h$, $b(\lambda^h, u^h, u^h) = 1$, such that

$$a_h(\lambda^h, u^h, v^h) = b_h(\lambda^h, u^h, v^h) \quad \forall v^h \in V_h. \quad (18)$$

For fixed $\mu \in \Lambda$, we introduce the following linear parametric variational eigenvalue problem: find the least $\gamma^h(\mu) \in \mathbb{R}$, $u^h \in K_h$, $b_h(\mu, u^h, u^h) = 1$, such that

$$a_h(\mu, u^h, v^h) = \gamma^h(\mu)b_h(\mu, u^h, v^h) \quad \forall v^h \in V_h. \quad (19)$$

Denote

$$R_h(\mu, v) = \frac{a_h(\mu, v^h, v^h)}{b_h(\mu, v^h, v^h)} \quad \forall v^h \in V_h \setminus \{0\}. \quad (20)$$

Then the following variational property is valid

$$\gamma^h(\mu) = \min_{v^h \in V_h \setminus \{0\}} R_h(\mu, v^h). \quad (21)$$

Let $\gamma_i^h(\mu)$, $u_i^h(\mu) = u_i^h(\mu, x)$, $x \in \bar{\Omega}$, $\mu \in \Lambda$, $i = 1, 2, \dots, N-1$, be eigenvalues and eigenfunctions satisfying equation (19),

$$\gamma_1^h(\mu) < \gamma_2^h(\mu) < \dots < \gamma_{N-1}^h(\mu), \quad (22)$$

$$a_h(\mu, u_i^h(\mu), u_j^h(\mu)) = \gamma_i^h(\mu)\delta_{ij}, \quad (23)$$

$$b_h(\mu, u_i^h(\mu), u_j^h(\mu)) = \delta_{ij}, \quad (24)$$

$i, j = 1, 2, \dots, N-1$, $u_i^h(\mu) \in K_h$, $\gamma_1^h(\mu) = \gamma^h(\mu)$, functions $u_i^h(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \dots, N-1$, form a complete system in the space V_h . The minimal eigenvalue λ^h of problem (18) is the least root of the equation $\gamma^h(\mu) = 1$, $\mu \in \Lambda$. Put $\sigma_h = (\sin(h/2)/(h/2))^2$.

Theorem 2. If $p(0)\sigma_h > r(0)$, then there exists a minimal simple eigenvalue of problem (18) corresponding to a positive eigenfunction.

Proof. Using the variational property of the minimal eigenvalue of problem (19), we derive

$$\gamma^h(0) = \min_{v^h \in V_h \setminus \{0\}} R_h(0, v^h) = \quad (25)$$

$$= \min_{v^h \in V_h \setminus \{0\}} \frac{\int_0^\pi p(0)((v^h(x))')^2 dx}{\int_0^\pi r(0)(v^h(x))^2 dx} = \quad (26)$$

$$= \frac{p(0)}{r(0)} \min_{v^h \in V_h \setminus \{0\}} \frac{\int_0^\pi ((v^h(x))')^2 dx}{\int_0^\pi (v^h(x))^2 dx} = \frac{p(0)}{r(0)} \sigma_h > 1. \quad (27)$$

Denote

$$s_1 = \min_{x \in \Omega} s(x), \quad p_1 = \sup_{\mu \in \Lambda} p(\mu). \quad (28)$$

Then we obtain

$$\gamma^h(\mu) = \min_{v^h \in V_h \setminus \{0\}} R_h(\mu, v^h) = \quad (29)$$

$$= \min_{v^h \in V_h \setminus \{0\}} \frac{\int_0^\pi p(\mu)((v^h(x))')^2 dx}{\int_0^\pi r(\mu)(v^h(x))^2 dx} \leq \quad (30)$$

$$\leq \frac{p_1}{r(\mu s_1)} \min_{v^h \in V_h \setminus \{0\}} \frac{\int_0^\pi ((v^h(x))')^2 dx}{\int_0^\pi (v^h(x))^2 dx} = \quad (31)$$

$$\leq \frac{p_1}{r(\mu s_1)} \sigma_h \rightarrow 0 \quad (32)$$

as $\mu \rightarrow \infty$, since $r(\mu) \rightarrow \infty$, as $\mu \rightarrow \infty$.

Thus, the continuous function $\gamma^h(\mu)$, $\mu \in \Lambda$, has the properties $\gamma^h(0) > 1$ and $\gamma^h(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$. This implies the existence of a minimal root of equation

$$\gamma^h(\mu) = 1, \quad \mu \in \Lambda, \quad (33)$$

which defines the minimal eigenvalue λ^h of problem (18). The minimal eigenvalue λ^h is simple and corresponds to a positive eigenfunction, because $\gamma^h(\mu)$

with $\mu \in \Lambda$ is the simple eigenvalue of the parametric problem (19) corresponding to a positive eigenfunction. This completes the proof of the theorem.

Denote $y_i = u^h(x_i)$, $i = 0, 1, \dots, N$,

$$\tilde{p}_i(\lambda^h) = p(\lambda^h s(x_i - h/2)), \quad (34)$$

$$r_i(\lambda^h) = r(\lambda^h s(x_i)), \quad (35)$$

$y_{x,i} = (y_{i+1} - y_i)/h$, $y_{\bar{x},i} = (y_i - y_{i-1})/h$. Then problem (18) can be written in the finite difference form

$$-(\tilde{p}(\lambda^h) y_{\bar{x}})_{x,i} = \lambda^h r_i(\lambda^h) y_i, \quad i = 1, 2, \dots, N-1, \quad (36)$$

$$y_0 = y_N = 0. \quad (37)$$

By c we denote various positive constants independent of h . Put

$$a'(\mu, u, v) = \int_0^\pi p'(\mu s(x)) s(x) u' v' dx, \quad (38)$$

$$b'(\mu, u, v) = \int_0^\pi r'(\mu s(x)) s(x) u v dx, \quad (39)$$

for $\mu \in \Lambda$, $u, v \in V$. Then the function $\gamma^h(\mu)$, $\mu \in \Lambda$, is continuous and the following formula is valid

$$\gamma'(\mu) = a'(\mu, v, v) - \gamma(\mu) b'(\mu, v, v) \quad (40)$$

for $\mu \in \Lambda$, $v = u_1(\mu)$.

Theorem 3. Suppose $\gamma'(\lambda) \neq 0$, λ , u , and λ^h , u^h , are eigenvalues and eigenfunctions of the eigenvalue problems (10) and (18). Then the estimates

$$|\lambda^h - \lambda| \leq ch^2, \quad |u^h - u|_1 \leq ch, \quad (41)$$

hold for sufficiently small h .

Proof. For $\mu \in \Lambda$, $v^h, w^h \in V_h$ we have the estimates

$$|a_h(\mu, v^h, w^h) - a(\mu, v^h, w^h)| \leq ch^2 |v^h|_1 |w^h|_1, \quad (42)$$

$$|b_h(\mu, v^h, w^h) - b(\mu, v^h, w^h)| \leq ch^2 |v^h|_1 |w^h|_1. \quad (43)$$

Then the error estimates of Theorem 3 are proved with using results from [19-23]. This completes the proof of the theorem.

To illustrate the theoretical results of Theorems 1 and 2, we have solved the eigenvalue problem (10) for $s(x) = \sqrt{\pi - x} + 1$, $x \in \bar{\Omega}$,

$$p(\mu) = \begin{cases} -\mu^3 + \mu^2 + \mu + 2, & \mu \in [0, 1], \\ 3, & \mu \in (1, \infty), \end{cases} \quad (44)$$

$$r(\mu) = \begin{cases} 1, & \mu \in [0, 1], \\ (\mu - 1)^2 / 2 + 1, & \mu \in (1, \infty), \end{cases} \quad (45)$$

using the finite difference scheme (36), (37) for $N = 100$, $\mu \in [0, 3]$. In this case, $p(0) = 2$, $r(0) = 1$, $\sigma_h = 0.99992$; therefore, the conditions of Theorems 1 and 2 are valid. Fig. 1 shows the graph of the function $\gamma(\mu)$ of the parametric eigenvalue problem (11) and the minimal simple eigenvalue $\lambda = 1.3248$ of the nonlinear eigenvalue problem (10). We see that the experimental results are consistent with the theoretical results in Theorems 1 and 2. Note that investigations of the present paper generalize well known results for eigenvalue problems with linear dependence on the spectral parameter.

Fig. 1. The minimal eigenvalue λ .

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