Eigenvibrations of a bar with load

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Abstract. The differential eigenvalue problem describing eigenvibrations of an elastic bar with load is investigated. The problem has an increasing sequence of positive simple eigenvalues with limit point at infinity. To the sequence of eigenvalues, there corresponds a complete orthonormal system of eigenfunctions. We formulate limit differential eigenvalue problems and prove the convergence of the eigenvalues and eigenfunctions of the initial problem to the corresponding eigenvalues and eigenfunctions of the limit problems as load mass tending to infinity. The original differential eigenvalue problem is approximated by the finite difference method on a uniform grid. Error estimates for approximate eigenvalues and eigenfunctions are established. Investigations of this paper can be generalized for the cases of more complicated and important problems on eigenvibrations of beams, plates and shells with attached loads.

1 Introduction

Let us formulate the differential eigenvalue problem governing longitudinal eigenvibrations of the bar-load system. Assume that the elastic bar axis occupies in the equilibrium position the segment $[0, l]$ on the Ox axis. Denote by $\rho(x)$ the volume mass density, $E(x)$ the elasticity Young modulus, $S(x)$ the square of transversal cut of the bar. Suppose that the end $x = 0$ of the bar is rigidly fixed while the end $x = l$ is free, at the point $x = l$ of the bar a load of mass M is rigidly joined. Then the longitudinal deflection $w(x, t)$ of the bar at a point x at time t satisfies the following equations

$$\frac{\partial}{\partial x} \left(p(x) \frac{\partial}{\partial x} w(x, t) \right) = r(x) \frac{\partial^2}{\partial t^2} w(x, t), \quad x \in (0, l), \quad (1)$$

$$w(0, t) = 0, \quad -p(l) \frac{\partial}{\partial x} w(l, t) = M \frac{\partial^2}{\partial t^2} w(l, t), \quad (2)$$

where $t > 0$, $p(x) = E(x)S(x)$, $r(x) = \rho(x)S(x)$, $x \in [0, l]$.

The eigenvibrations of the bar-load system are characterized by the function $w(x, t)$ of the following form $w(x, t) = u(x)v(t)$, $x \in [0, l]$, where $v(t) = a_0 \cos \sqrt{\lambda} t + b_0 \sin \sqrt{\lambda} t$, $t > 0$; a_0 , b_0 , λ are constants. Then equations (1) and (2) lead to the following differential eigenvalue problem of second order: find numbers λ and nontrivial functions $u(x)$, $x \in [0, l]$, such that

$$-(p(x)u'(x))' = \lambda r(x)u(x), \quad x \in (0, l), \quad (3)$$

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$$u(0) = 0, \quad p(l)u'(l) = \lambda Mu(l). \quad (4)$$

Problem (3), (4) has an increasing sequence of positive simple eigenvalues with limit point at infinity. To the sequence of eigenvalues, there corresponds a complete orthonormal system of eigenfunctions. In the present paper, limit properties as $M \rightarrow \infty$ of eigenvalues and eigenfunctions of parameter dependent eigenvalue problem (3), (4) with parameter M are studied.

Parameter eigenvalue problems are applied for the investigation and solution of eigenvalue problems with nonlinear dependence on the spectral parameter. Nonlinear eigenvalue problems arise in various applications [1-3]. Numerical methods for solving matrix eigenvalue problems with nonlinear dependence on the spectral parameter were constructed and investigated in the papers [4-13]. Mesh methods for solving differential nonlinear eigenvalue problems were studied in the papers [14-16]. The theoretical basis for the study of nonlinear spectral problems is results obtained for linear eigenvalues problems [17-23]. In the papers [24-30], numerical methods for solving applied nonlinear boundary value problems and variational inequalities have been studied.

2 Variational statement of the problem

Let $\mathbb{R}$ be the real line, $\Omega = (0, l)$, $\bar{\Omega} = [0, l]$, $\Lambda = (0, \infty)$. As usual, we denote by $L_2(\Omega)$ and $W_2^1(\Omega)$ the real Lebesgue space and the real Sobolev space, respectively, with norms $|\cdot|_0$ and $\|\cdot\|_1$, where

$$|v|_0 = \left(\int_0^l (v(x))^2 dx \right)^{1/2}, \quad (5)$$

$$\|v\|_1 = (\|v\|_0^2 + |v|_1^2)^{1/2}, \quad |v|_1 = |v'|_0. \quad (6)$$

Set $H = L_2(\Omega)$, $V = \{v: v \in W_2^1(\Omega), v(0) = 0\}$. For a continuous function $v(x)$, $x \in \bar{\Omega}$, we denote

$$|v|_{0,\infty} = \max_{x \in \bar{\Omega}} |v(x)|. \quad (7)$$

Note that the space V is compactly embedded into the space H , any function from V is continuous on $\bar{\Omega}$. The semi-norm $|\cdot|_1$ is a norm over the space V , which is equivalent to the norm $\|\cdot\|_1$. There exist constants c_0 and c_1 such that $|v|_{0,\infty} \leq c_0 |v|_1$, $|v|_0 \leq c_1 |v|_1$, for any $v \in V$.

Define sufficiently smooth functions $p(x)$, $r(x)$, $x \in \bar{\Omega}$, for which there exist positive constants p_i , r_i , $i=1,2$, such that $p_1 \leq p(x) \leq p_2$, $r_1 \leq r(x) \leq r_2$. Introduce a number $\mu \in \Lambda$ and bilinear forms $a: V \times V \rightarrow \mathbb{R}$, $b: V \times V \rightarrow \mathbb{R}$, $c: V \times V \rightarrow \mathbb{R}$, by the formulas

$$a(u, v) = \int_0^l p(x) u' v' dx, \quad b(u, v) = \int_0^l r(x) u v dx, \quad (8)$$

$$c(u, v) = u(l)v(l), \quad \text{for any } u, v \in V.$$

For $\mu = M$, the differential problem (3), (4) is equivalent to the following variational eigenvalue problem: find $\lambda \in \mathbb{R}$, $u \in V \setminus \{0\}$, such that

$$a(u, v) = \lambda(b(u, v) + \mu c(u, v)) \quad \forall v \in V. \quad (9)$$

The number $\lambda = \varphi(\mu)$ that satisfies (9) is called an eigenvalue, and the function $u = u^\mu$ is called an eigenfunction of problem (9) corresponding to λ . The set $U(\lambda)$ that consists of the eigenfunctions corresponding to the eigenvalue λ and the zero function is a closed subspace in V , which is called the eigensubspace corresponding to the eigenvalue λ . The dimension of this subspace is called the multiplicity of the eigenvalue λ . If the dimension of an eigensubspace is equal to unity, then the corresponding eigenvalue is said to be simple. The pair (λ, u) with λ and u satisfying (9) is called an eigensolution or eigenpair of problem (9).

3 Parameter eigenvalue problem

Let us formulate the following existence results for problem (9). Set

$$R(v) = R_\mu(v) = \frac{a(v, v)}{b(v, v) + \mu c(v, v)} \quad (10)$$

for any $v \in V \setminus \{0\}$, $\mu \in \Lambda$. If W is a subspace of the space V , then we denote

$$W_a^\perp = \{v: v \in V, a(v, w) = 0 \quad \forall w \in W\}. \quad (11)$$

Denote by $S_k(W)$ the set of all k -dimensional subspaces of the space W for $k \geq 1$. The set $S_0(W)$ consists of $\{0\}$ alone. Put $S_k = S_k(V)$ for $k \geq 0$.

Problem (9) has an increasing sequence of positive simple eigenvalues with limit point at infinity $\lambda_k = \varphi_k(\mu)$, $k = 0, 1, \dots$, $0 < \lambda_0 < \lambda_1 < \dots < \lambda_k < \dots$, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. These eigenvalues correspond to a complete orthonormal system of eigenfunctions $u_k = u_k^\mu$, $k = 0, 1, \dots$, which can be chosen according to one of the conditions:

$$\begin{aligned} \text{a)} \quad & a(u_i, u_j) = \delta_{ij}, \quad b(u_i, u_j) + \mu c(u_i, u_j) = \delta_{ij} / \lambda_i, \\ & i, j = 0, 1, \dots; \\ \text{b)} \quad & a(u_i, u_j) = \lambda_i \delta_{ij}, \quad b(u_i, u_j) + \mu c(u_i, u_j) = \delta_{ij}, \\ & i, j = 0, 1, \dots \end{aligned}$$

Functions u_k , $k = 0, 1, \dots$, form a complete system in the space V . The following relations hold: $u_k(l) \neq 0$, $k = 0, 1, \dots$. The following variational properties are valid:

$$\lambda_{k-1} = \min_{v \in (E_{k-1})_a^\perp \setminus \{0\}} R(v) = \max_{v \in E_k \setminus \{0\}} R(v), \quad (12)$$

$$\lambda_{k-1} = \max_{W \in S_{k-1}} \min_{v \in W_a^\perp \setminus \{0\}} R(v) = \min_{W \in S_k} \max_{v \in W \setminus \{0\}} R(v), \quad (13)$$

where $E_k = E_k^\mu = \text{span}\{u_0, u_1, \dots, u_{k-1}\}$, $k = 1, 2, \dots$, $E_0 = E_0^\mu = \{0\}$, $(E_0)_a^\perp = V$.

The following inequalities hold:

$$\alpha_1 |v|_1^2 \leq a(v, v) \leq \alpha_2 |v|_1^2, \quad (14)$$

$$\beta_1 |v|_0^2 \leq b(v, v) \leq \beta_2 |v|_0^2, \quad (15)$$

$$0 \leq c(v, v) \leq \gamma_2 |v|_1^2, \quad \text{for any } v \in V, \quad \alpha_i = p_i, \quad \beta_i = r_i, \\ i = 1, 2, \quad \gamma_2 = c_0^2.$$

The functions $\varphi_k(\mu)$, $\mu \in \Lambda$, $k = 0, 1, \dots$, are continuous and decreasing.

4 Limit properties of parameter problem

Define the subspaces $V_0 = \{v: v \in W_2^1(\Omega), v(0) = 0, v(l) = 0\}$ and $V_1 = (V_0)_a^\perp$ of the space V . Introduce the following linear eigenvalue problems.

Find $\lambda^{(0)} \in \mathbb{R}$, $u \in V_0 \setminus \{0\}$, such that

$$a(u, v) = \lambda^{(0)} b(u, v) \quad \forall v \in V_0. \quad (16)$$

Find $\lambda^{(1)} \in \mathbb{R}$, $u \in V_1 \setminus \{0\}$, such that

$$a(u, v) = \lambda^{(1)} c(u, v) \quad \forall v \in V_1. \quad (17)$$

Problem (16) has an increasing sequence of positive simple eigenvalues with limit point at infinity $\lambda_k^{(0)}$,

$k=1,2,\dots$, $0 < \lambda_1^{(0)} < \lambda_2^{(0)} < \dots < \lambda_k^{(0)} < \dots$, $\lambda_k^{(0)} \rightarrow \infty$ as $k \rightarrow \infty$. The corresponding eigenfunctions $u_k^{(0)}$, $k=1,2,\dots$, form a complete orthonormal system in the space V_0 .

Since $\dim V_1 = \text{codim } V_0 = 1$, the eigenvalue problem (17) has one eigenvalue $\lambda^{(1)}$ which is positive and simple. There exists a unique normalized eigenfunction $u^{(1)}$ such that $a(u^{(1)}, u^{(1)}) = 1$, $u^{(1)}(l) > 0$, corresponding to the eigenvalue $\lambda^{(1)}$. Moreover, the functions $u_0^{(0)} = u^{(1)}$, $u_k^{(0)}$, $k=1,2,\dots$, form a complete orthonormal system in V such that $a(u_i^{(0)}, u_j^{(0)}) = \delta_{ij}$, $i, j = 0, 1, \dots$. The eigenvalue and the eigenfunction of the eigenvalue problem (17) are defined by

$$\lambda^{(1)} = \left(\int_0^l p(y)^{-1} dy \right)^{-1}, \quad (18)$$

$$u^{(1)}(x) = \int_0^x p(y)^{-1} dy \left(\int_0^l p(y)^{-1} dy \right)^{-1/2}, \quad x \in [0, l]. \quad (19)$$

Theorem 1. The convergence $\varphi_0(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$ holds, and the following asymptotic representation is valid $\varphi_0(\mu) = \lambda^{(1)} \mu^{-1} + o(\mu^{-1})$ as $\mu \rightarrow \infty$. Moreover, the convergence $u_0^\mu \rightarrow u^{(1)}$ in V as $\mu \rightarrow \infty$ holds, where $a(u_0^\mu, u_0^\mu) = 1$, $u_0^\mu(l) > 0$, $\mu \in \Lambda$, $a(u^{(1)}, u^{(1)}) = 1$, $u^{(1)}(l) > 0$.

Theorem 2. The convergence $\varphi_k(\mu) \rightarrow \lambda_k^{(0)}$, $u_k^\mu \rightarrow u_k^{(0)}$ in V , $u_k^\mu(l) = O(\mu^{-1})$, as $\mu \rightarrow \infty$ holds, where $a(u_k^\mu, u_k^\mu) = 1$, $u_k^\mu(l) > 0$, $\mu \in \Lambda$, $a(u_k^{(0)}, u_k^{(0)}) = 1$, $a(u_k^{(0)}, u_k^\mu) > 0$, $k=1,2,\dots$.

Theorems 1 and 2 are proved with using results from [15, 16].

5 Finite difference approximation of the problem

Let us partition the interval $[0, l]$ by equidistant points $x_i = ih$, $i = 0, 1, \dots, N$, into elements $e_i = [x_{i-1}, x_i]$, $i = 1, 2, \dots, N$, $h = l/N$, and let V_h denote the subspace of the space V consisting of continuous functions v^h linear on each element e_i , $i = 1, 2, \dots, N$. For $u^h, v^h \in V_h$, we introduce approximate bilinear forms

$$a_h(u^h, v^h) = \sum_{i=1}^N h p(x_i - h/2) (u^h(x_i - h/2))' (v^h(x_i - h/2))', \quad (20)$$

$$b_h(u^h, v^h) = \sum_{i=1}^N h (r(x_{i-1}) u^h(x_{i-1}) v^h(x_{i-1}) + r(x_i) u^h(x_i) v^h(x_i)) / 2. \quad (21)$$

We approximate the original eigenvalue problem (9) by a finite-dimensional eigenvalue problem: find $\lambda^h \in \mathbb{R}$, $u^h \in V_h \setminus \{0\}$, such that

$$a_h(u^h, v^h) = \lambda^h (b_h(u^h, v^h) + \mu c(u^h, v^h)) \quad \forall v^h \in V_h. \quad (22)$$

Problem (22) has an increasing finite sequence of positive simple eigenvalues λ_k^h , $k = 0, 1, \dots, N-1$, $0 < \lambda_0^h < \lambda_1^h < \dots < \lambda_{N-1}^h$. These eigenvalues correspond to an orthonormal system of eigenfunctions u_k^h , $k = 0, 1, \dots, N-1$, which can be chosen according to one of the conditions:

$$a^h) \quad a_h(u_i^h, u_j^h) = \delta_{ij}, \quad b_h(u_i^h, u_j^h) + \mu c(u_i^h, u_j^h) = \delta_{ij} / \lambda_i^h, \quad i, j = 0, 1, \dots, N-1;$$

$$b^h) \quad a_h(u_i^h, u_j^h) = \lambda_i^h \delta_{ij}, \quad b_h(u_i^h, u_j^h) + \mu c(u_i^h, u_j^h) = \delta_{ij}, \quad i, j = 0, 1, \dots, N-1.$$

Functions u_k^h , $k = 0, 1, \dots, N-1$, form a complete orthonormal system in the space V_h .

Denote $y_i = u^h(x_i)$, $i = 0, 1, \dots, N$, $\tilde{p}_i = p(x_i - h/2)$, $r_i = r(x_i)$, $y_{x,i} = (y_{i+1} - y_i)/h$, $y_{\bar{x},i} = (y_i - y_{i-1})/h$. Then problem (22) can be written in the following finite difference form

$$-(\tilde{p} y_{\bar{x}})_{x,i} = \lambda^h r_i y_i, \quad i = 1, 2, \dots, N-1, \quad (23)$$

$$y_0 = 0, \quad \tilde{p}_N y_{\bar{x},N} = \lambda^h h \left(\frac{1}{2} r_N + \frac{\mu}{h} \right) y_N. \quad (24)$$

By c we denote various positive constants independent of h .

Theorem 3. Suppose that conditions b) and b^h) are valid. Then the estimates $|\lambda_k^h - \lambda_k| \leq ch^2$, $\|u_k^h - u_k\| \leq ch$, hold for sufficiently small h , $a(u_k^h, u_k) > 0$.

Proof. For any $v^h, w^h \in V_h$ we have the following estimates

$$|a_h(v^h, w^h) - a(v^h, w^h)| \leq ch^2 \|v^h\|_1 \|w^h\|_1, \quad (25)$$

$$|b_h(v^h, w^h) - b(v^h, w^h)| \leq ch^2 \|v^h\|_1 \|w^h\|_1. \quad (26)$$

Now desired error estimates are proved with using results from [19-23].

To illustrate theoretical results of Theorems 1 and 2, we have solved the eigenvalue problem (9) for $p(x) = 1$, $r(x) = 1$, $l = 1$, using the finite difference scheme (23), (24) for $N = 100$, $\mu \in [0, 2]$. Fig. 1 shows the graphs of the functions $\lambda_k = \varphi_k(\mu)$, $\mu \in [0, 2]$, $k = 0, 1, \dots, 5$, and

the eigenvalues $\lambda_k^{(0)} = (k\pi)^2$, $k = 1, 2, \dots, 5$, of the limit eigenvalue problem (16). We see that the experimental results are consistent with the theoretical results in Theorems 1 and 2; namely, $\phi_0(\mu) \rightarrow 0$, $\phi_k(\mu) \rightarrow \lambda_k^{(0)}$, as $\mu \rightarrow \infty$, $k = 1, 2, \dots, 5$. Note that investigations of the present paper can be generalized for the cases of more complicated and important problems on eigenvibrations of beams, plates and shells with attached loads.

Fig.1. Limit properties of the eigenvalues $\lambda_k = \phi_k(\mu)$, $\mu \in [0, 2]$, $k = 0, 1, \dots, 5$.

This work was supported by Russian Science Foundation, project no. 16-11-10299.

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