Energy method in solving the problems of stability for a viscoelastic polymer rods

Serdar Yazyev^{1,*}, Maria Kozelskaya¹, Grigory Strelnikov¹, and Stepan Litvinov¹

¹ Don State Technical University, 344022 Sotcialisticheskaya 162, Rostov-on-Don, Russia

Abstract. With the energy method in the form of Ritz-Timoshenko solved the stability problem for a polymer rod under axial compression for the clamping-free edge option. The proposed form of loss of stability is chosen as a sum of functions with indeterminate coefficients. The shape functions for various fastenings and the represented fastening of the rod in particular are considered. The result is obtained numerically using the MatLab complex. A study was made of the long critical loads. It is shown that if the compressive force does not exceed the long-term critical one, then stability loss does not occur.

1 Introduction

Analysis of the load-carrying capacity of the structure, in addition to strength calculation, should include the stability issues of the entire system and its individual elements. This is important for construction, mechanical engineering, since many structural elements work on compression and can lose stability at stresses much less than destructive [1-4]

To reduce weight, cost and labor, and also to improve the quality of building structures and their elements, it is necessary to develop and introduce new types of light and lightweight structures. A significant role in the development of effective designs is performed by improving the calculation methods taking into account the actual work of the material [9-12].

One of the factors that have a significant effect on the stress-strain state and the deformability of the structures is creep.

The aim of the article is to solve the stability problems of rods, taking into account initial imperfections, various variants of fixing, and arbitrary creep laws.

2 Formulation of the problem

According to the Ritz-Timoshenko method, the assumed form of loss of stability is chosen as a sum of functions with indeterminate coefficients a_i ($i = 1, \dots, n$):

$$v(x) = \sum_{i=1}^n a_i f_i \quad (1)$$

As f_i we take functions from x , satisfying the kinematic boundary conditions of the problem, i.e. those that relate to the deflection and rotation angles, regardless of a_i .

* Corresponding author: ps62@yandex.ru

The resolving equations for determining the unknown coefficients of the series are obtained from the condition of the minimum of the total potential energy of the $\mathcal{E}$ system:

$$\frac{\partial \mathcal{E}}{\partial a_1} = 0, \frac{\partial \mathcal{E}}{\partial a_2} = 0, \dots, \frac{\partial \mathcal{E}}{\partial a_n} = 0 \quad (2)$$

The total potential energy of the rod is written in the form:

$$\mathcal{E} = \Pi - A, \quad (3)$$

Where Π – potential energy of deformation,

$A = F\Delta$ – work of external forces.

The potential strain energy has the form:

$$\Pi = \frac{1}{2} \int_V \sigma \varepsilon^{el} dV = \frac{1}{2} \int_V E (\varepsilon^{el})^2 dV. \quad (4)$$

Here ε^{el} is meant an elastic deformation that represents the difference between total deformation and creep strain:

$$\varepsilon^{el} = \varepsilon_0 - \frac{d^2 v}{dx^2} y - \varepsilon^* \quad (5)$$

3 The derivation of the resolving equations

Substituting (5) in (4), we obtain:

$$\Pi = \frac{1}{2} \int_0^l E(x) dx \left\{ \begin{aligned} &\varepsilon_0^2 + \left(\frac{d^2 v}{dx^2} \right)^2 y^2 + (\varepsilon^*)^2 + \\ &+ 2\varepsilon^* \frac{d^2 v}{dx^2} y - 2\varepsilon_0 \varepsilon^* \\ &- 2\varepsilon_0 \frac{d^2 v}{dx^2} y \end{aligned} \right\} dA =$$

$$= \frac{1}{2} \int_0^l E(x) \left\{ \begin{aligned} &I(x) \left(\frac{d^2 v}{dx^2} \right)^2 + 2 \frac{d^2 v}{dx^2} \int_A \varepsilon^* y dA + \\ &+ \int_A (\varepsilon^*)^2 dA - 2\varepsilon_0 \int_A \varepsilon^* dA + \varepsilon_0^2 A \end{aligned} \right\} dx.$$

Approximation of the ends of the elementary fragment dx of the rod (see Figure 1) during buckling can be determined by formula:

$$\Delta(dx) = dx \cdot \left(\cos \left(\frac{dv_0}{dx} \right) - \cos \left(\frac{dv}{dx} + \frac{dv_0}{dx} \right) \right) =$$

$$-dx \cdot 2 \sin \frac{\frac{dv_0}{dx} + \left(\frac{dv}{dx} + \frac{dv_0}{dx} \right)}{2} \cdot \sin \frac{\frac{dv_0}{dx} - \left(\frac{dv}{dx} + \frac{dv_0}{dx} \right)}{2} =$$

$$= 2 \cdot \frac{\left(\left(2 \cdot \frac{dv_0}{dx} + \frac{dv}{dx} \right) \cdot \left(\frac{dv}{dx} \right) \right)}{4} = \left(\frac{1}{2} \left(\frac{dv}{dx} \right)^2 + \frac{dv}{dx} \frac{dv_0}{dx} \right) dx.$$

Fig. 1. To the determination of the approach of the ends of the rod during buckling.

Then the convergence of the ends of the whole rod will be written in the form:

$$\Delta = \frac{1}{2} \int_0^l \left\{ \left(\frac{dv}{dx} \right)^2 + 2 \frac{dv}{dx} \frac{dv_0}{dx} \right\} dx$$

The work of external forces is defined as follows:

$$A = F\Delta = \frac{1}{2} F \int_0^l \left\{ \left(\frac{dv}{dx} \right)^2 + 2 \frac{dv}{dx} \frac{dv_0}{dx} \right\} dx.$$

The total potential energy of the system is written in the form:

$$\Theta = \frac{1}{2} \int_0^l E(x) \left\{ \begin{aligned} &I(x) \left(\frac{d^2 v}{dx^2} \right)^2 + 2 \frac{d^2 v}{dx^2} \int_A \varepsilon^* y dA \\ &+ \int_A (\varepsilon^*)^2 dA - 2\varepsilon_0 \int_A \varepsilon^* dA + \varepsilon_0^2 A \end{aligned} \right\} dx -$$

$$- \frac{1}{2} F \int_0^l \left\{ \left(\frac{dv}{dx} \right)^2 + 2 \frac{dv}{dx} \frac{dv_0}{dx} \right\} dx. \quad (6)$$

After substituting (1) in (6) and then minimizing it with unknown coefficients a_i we obtain:

$$\frac{\partial \Theta}{\partial a_j} = \sum_{i=1}^n a_i \left\{ \begin{aligned} &\int_0^l E(x) I(x) \frac{d^2 f_i}{dx^2} \frac{d^2 f_j}{dx^2} dx \\ &- F \int_0^l \frac{df_i}{dx} \frac{df_j}{dx} dx \end{aligned} \right\} +$$

$$+ \int_0^l E(x) \frac{d^2 f_j}{dx^2} \left\{ \int_A \varepsilon^* y dA \right\} dx -$$

$$- F \int_0^l \frac{df_j}{dx} \frac{dv_0}{dx} dx = 0, \quad j = 1 \dots n$$

Or in the matrix form:

$$([K] - F[B])\{X\} = \{C\}, \quad (7)$$

$$\text{Where } X = \begin{Bmatrix} a_1 \\ \dots \\ a_n \end{Bmatrix}, \quad \{C\} = \begin{Bmatrix} c_1 \\ \dots \\ c_n \end{Bmatrix}, \quad K = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}.$$

$$k_{ij} = \int_0^l E(x) I(x) \frac{d^2 f_i}{dx^2} \frac{d^2 f_j}{dx^2} dx;$$

$$b_{ij} = \int_0^l \frac{df_i}{dx} \frac{df_j}{dx} dx;$$

$$c_j = - \int_0^l E(x) \frac{d^2 f_j}{dx^2} \left\{ \int_A \varepsilon^* y dA \right\} dx + F \int_0^l \frac{df_j}{dx} \frac{dv_0}{dx} dx.$$

Instant critical force can be determined from the condition that the determinant of the system (7):

$$\det([K] - F[B]) = 0. \quad (8)$$

Thus, the problem of determining the critical force reduces to a generalized vector equation. In the case where the matrix $[B]$ is not degenerate, system (7) can be multiplied by $[B]^{-1}$:

$$([B]^{-1}[K] - F[E])\{X\} = [B]^{-1}\{C\}, \quad (9)$$

where $[E]$ – unit matrix.

The first critical force will correspond to the minimum of the eigenvalues of the matrix $[B]^{-1}[K]$.

4 Method of solving the problem

Consider the function of the form $f_i(x)$ when fixing the rod embodiment "jamming - free edge" (see Figure 2). The initial deflection is defined as follows:

$$v_0(x) = f_0 \left(1 - \cos \left(\frac{\pi x}{2l} \right) \right). \quad (10)$$

The function takes the form of:

$$f_i(x) = 1 - \cos \left(\frac{\pi xi}{2l} \right). \quad (11)$$

The derivatives of (10) and (11) are written in the form:

$$\frac{dv_0}{dx} = \frac{\pi f_0}{2l} \sin \frac{\pi x}{2l}; \quad (12)$$

$$\frac{df_i}{dx} = \frac{\pi i}{2l} \sin \frac{\pi xi}{2l}; \quad (13)$$

$$\frac{d^2 f_i}{dx^2} = \frac{\pi^2 i^2}{4l^2} \cos \frac{\pi xi}{2l}.$$

Fig. 2. The option of embodiment a "jamming-free edge."

The integrals in the formulas of the coefficients of the system of equations are written in the form:

$$\int_0^l \frac{df_i}{dx} \frac{df_j}{dx} dx = \left(\frac{\pi}{2l} \right)^2 ij \int_0^l \sin \left(\frac{\pi xi}{2l} \right) \cdot \sin \left(\frac{\pi xj}{2l} \right) dx = \begin{cases} 0, npu \ i \neq j; \\ \frac{\pi^2 ij}{8l}, npu \ i = j. \end{cases} \quad (14)$$

$$\int_0^l \frac{df_j}{dx} \frac{dv_0}{dx} dx = \frac{\pi^2 j f_0}{4l^2} \int_0^l \sin \left(\frac{\pi xj}{2l} \right) \cdot \sin \left(\frac{\pi x}{2l} \right) dx = \begin{cases} 0, npu \ j \neq 1; \\ \frac{\pi^2 j f_0}{8l}, npu \ j = 1. \end{cases} \quad (15)$$

In the case of constant stiffness of the rod k_{ij} the coefficients take the form:

$$k_{ij} = EI \int_0^l \frac{d^2 f_i}{dx^2} \frac{d^2 f_j}{dx^2} dx = EI \left(\frac{\pi}{2l} \right)^4 i^2 j^2 \cdot \int_0^l \cos \left(\frac{\pi xi}{2l} \right) \cos \left(\frac{\pi xj}{2l} \right) dx = \begin{cases} 0, npu \ i \neq j; \\ \frac{EI \pi^4 i^2 j^2}{32l^3}, npu \ i = j. \end{cases} \quad (16)$$

5 Results and discussion

We consider a rod of rectangular constant cross-section from epoxy resin EDT-10. Rod sizes: $l = 157 \text{ мм}$, $b = 15 \text{ мм}$, $h = 8 \text{ мм}$, $f_0 = 0,16 \text{ мм}$. The rod is acted upon by force $F = 0,165 \text{ КН}$. This problem was solved by M. Y. Kozelskaya in [2], but by a different method. She received the critical time value - $t_{kp} = 1443 \text{ мин}$.

The Euler force for the problem under consideration: $F_s = \frac{\pi^2 EI}{4l^2} = 0,189 \text{ КН}$. Prolonged critical

force: $F_d = \frac{H}{E} F_s = 0,097 \text{ КН}$.

Figure 3 shows a graph of deflection height. Critical time $t_{kp} = 1441 \text{ мин}$. The rod loses stability over 130 hours. For a criterion of stability loss was taken condition: $f > \frac{l}{10}$.

Fig. 3. The growth of the deflection at $F = 0,165 \text{ КН}$

This problem was solved both by the finite difference method and by the Ritz-Timoshenko method.

Comparison of the value of the deflection at different times is shown in table 1.

Table 1. Comparison of deflections of the rod, obtained by the Ritz-Timoshenko method and using the finite difference method.

t, hour	20	40	60	80	100	120	130
f, mm							
Ritz-Timoshenko method	1.5 425	2.1 131	2.8 719	3.9 430	5.6 411	9.2 379	14. 930
Finite difference method	1.5 439	2.1 151	2.8 747	3.9 472	5.6 480	9.2 533	14. 989

The change in deflection as a function of x and t is shown in Fig. 4.

When solving the Ritz-Timoshenko method, the number of terms in the series n assumed to be equal to 4. The number of intervals for x and y is equal to 50.

From the table presented, it can be seen that the results are the same, which indicates their reliability, as well as the reliability of the calculation methodology developed by the authors.

Fig. 4. The growth of the deflection at $F=0.165 \text{ kH}$

The test problem was also solved for $F = 80H$, which is less than $F_{0.1}$. The theoretical value of the deflection at the end of the creep process:

$$f_{\infty} = \frac{0.16}{97.6/80 - 1} = 0.727 \text{ mm.}$$
 The growth curve of the deflection when $F = 80H$ is shown in Figure 5. At $t = 15000 \text{ h}$, $f = 0.726$.

Fig. 5. The growth of deflection at $F=80 \text{ H}$.

6 Conclusions

The resolving equations are obtained and a method based on the finite difference method is developed, the Ritz-Timoshenko method for calculating the stability of creep rods of variable stiffness with allowance for initial imperfections, various fixing variants, and arbitrary creep laws.

For compressed polymer rods that obey the nonlinear Maxwell-Gurevich equation, a technique has been developed for determining long-term critical loads. It is established that if the compressive does not exceed a prolonged critical one, then stability loss does not occur.

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