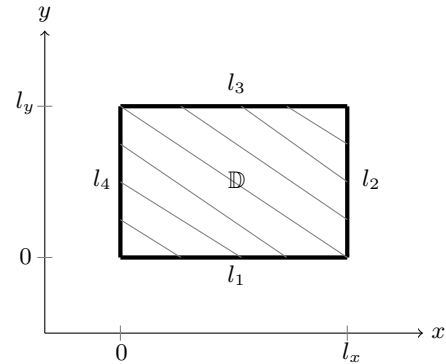


Mean Square Consistency On Numerical Solutions of Stochastic Wave Equation with Cubic Nonlinearities on 2D Rectangles

Haziem M. Hazaimeh

Department of Mathematics and statistics, Zayed University, Dubai, UAE

Abstract—Abstract. In this article we study the mean square consistency on numerical solutions of stochastic wave equations with cubic nonlinearities on two dimensional rectangles. In [8], we proved that the strong Fourier solution of these semi-linear wave equations exists and is unique on an appropriate Hilbert space. A linear-implicit Euler method is used to discretize the related Fourier coefficients. We prove that the linear-implicit Euler method applied to a solution of nonlinear stochastic wave equations in two dimensions is mean square consistency under the geometric condition.



I. INTRODUCTION

In this article we study the linear-implicit Euler method for the numerical solution of semi-linear stochastic wave equations

$$u_{tt} = \sigma^2 \Delta u + A(u, u_t) + B(u, u_t) \frac{dW}{dt}$$

with cubic nonlinearities in two dimensions in terms of all systems parameters, i.e., with non-global Lipschitz continuous nonlinearities. Our study focuses on numerical solution using linear-implicit Euler method (LIEM) under the geometric condition

$$\sigma^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right) - a_1 = \frac{\sigma^2 \pi^2 (l_x^2 + l_y^2) - a_1 l_x^2 l_y^2}{l_x^2 l_y^2} := \gamma > 0,$$

where $0 \leq x \leq l_x$ and $0 \leq y \leq l_y$ such that $\mathbb{D} = [0, l_x] \times [0, l_y]$. (Note that l_x and l_y are the dimension parameters of a vibrating plate or a membrane). We shall impose the following boundary conditions: (note that $t > 0$)

TABLE I: Boundary Conditions

l_1	$u(x, 0, t) = 0$	or	$u_x(x, 0, t) = 0$	or	$u_y(x, 0, t) = 0$
l_2	$u(l_x, y, t) = 0$	or	$u_x(l_x, y, t) = 0$	or	$u_y(l_x, y, t) = 0$
l_3	$u(x, l_y, t) = 0$	or	$u_x(x, l_y, t) = 0$	or	$u_y(x, l_y, t) = 0$
l_4	$u(0, y, t) = 0$	or	$u_x(0, y, t) = 0$	or	$u_y(0, y, t) = 0$

where l_1, l_2, l_3 , and l_4 as in the following figure

Also the initial conditions are $u(x, y, 0) = f(x, y)$ with $f \in \mathbb{L}^2$ (initial position) and $u_t(x, y, 0) = g(x, y)$ with $g \in \mathbb{L}^2$ (initial velocity). Recall that

$$\mathbb{L}^2(\mathbb{D}) = \{f : \mathbb{D} \rightarrow \mathbb{R} \mid \int_{\mathbb{D}} |f(x, y)|^2 d\mu(x, y) < \infty\},$$

where μ is the Lebesgue measure in two dimensions.

There is a little information about the numerical solution of nonlinear stochastic wave equations. However, Schurz [17] proved that the linear-implicit method for the nonlinear stochastic heat equation just in \mathbb{R} is mean square consistent with rate $r_0 = 1.5$. Also, Higham [11] studied the mean square stability of stochastic theta method and plotted the mean square stability when the test equation has real parameters. Higham, D.J., Mao, X., Stuart, A.M. [12] proved that strong convergence results but they took less restrictive conditions. To my information and since there are a few researcher work on mean square consistent of LIEM of nonlinear stochastic wave equations in two dimensions, which make me interested in working that.

Consider the stochastic nonlinear wave equation with both additive and multiplicative noise

$$u_{tt} = \sigma^2(u_{xx} + u_{yy}) + (a_1 - a_2 \|u\|_{\mathbb{L}^2(\mathbb{D})}^2)u - \kappa v + (b_0 + b_1 \|u\|_{\mathbb{L}^2(\mathbb{D})} + b_2 \|v\|_{\mathbb{L}^2(\mathbb{D})}) \frac{dW}{dt} \quad (1)$$

with $v = u_t$, where

$$W(x, y, t) = \sum_{i,j=1}^2 \sum_{n,m=1}^{\infty} \alpha_{n,m}^{i,j} W_{n,m}^{i,j}(t) e_n^i(x) e_m^j(y)$$

and $e_n^{i,j}$ and $e_m^{i,j}$ are orthonormalized eigenfunctions of Laplace operator Δ on $\mathbb{D} = [0, l_x] \times [0, l_y]$, driven by *i.i.d.* standard Wiener processes $W_{n,m}^{i,j}$ with $\mathbb{E}[W_{n,m}^{i,j}(t)] = 0$, $\mathbb{E}[W_{n,m}^{i,j}(t)]^2 = t$. Recall

that $e_n^i(z)$ and $e_m^j(z)$ are one of the functions $\sqrt{\frac{2}{l_z}} \sin(k\pi z/l_z)$ or $\sqrt{\frac{2}{l_z}} \cos(k\pi z/l_z)$, with $z = x$ or y , $k = n$ or m , and $i, j = 1, 2$. The solution of equation (1) in term of Fourier series is

$$u(x, y, t) = \sum_{i,j=1}^2 \sum_{n,m=1}^N c_{n,m}^{i,j}(t) e_n^i(x) e_m^j(y). \quad (2)$$

The form of Fourier solutions u and its approximate Fourier solutions u_N given by

$$u_N(x, y, t) = \sum_{i,j=1}^2 \sum_{n,m=1}^N c_{n,m}^{i,j}(t) e_n^i(x) e_m^j(y) \quad (3)$$

with its coefficients $c_{n,m}^{i,j}$ satisfying

$$\begin{aligned} \dot{c}_{n,m}^{i,j}(t) &= [-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \|u\|_{\mathbb{L}^2(\mathbb{D})}^2] c_{n,m}^{i,j}(t) \\ &\quad - \kappa v_{n,m}^{i,j}(t) + (b_0 + b_1 \|u\|_{\mathbb{L}^2(\mathbb{D})} \\ &\quad + b_2 \|\dot{u}\|_{\mathbb{L}^2(\mathbb{D})}) \alpha_{n,m}^{i,j} \dot{W}_{n,m}^{i,j}(t) \end{aligned} \quad (4)$$

where $\lambda_n = \frac{\pi^2 n^2}{l_x^2}$, $\beta_m = \frac{\pi^2 m^2}{l_y^2}$, $\|u\|_{\mathbb{L}^2(\mathbb{D})} =$

$$\sqrt{\sum_{i,j=1}^2 \sum_{n,m=1}^N [c_{n,m}^{i,j}]^2}, \text{ and } \|\dot{u}\|_{\mathbb{L}^2(\mathbb{D})} = \sqrt{\sum_{i,j=1}^2 \sum_{n,m=1}^N [\dot{c}_{n,m}^{i,j}]^2}. \text{ Note}$$

that equation (4) is equivalent to the following system

$$\begin{aligned} \dot{c}_{n,m}^{i,j} &= v_{n,m}^{i,j} \\ \dot{v}_{n,m}^{i,j} &= \left[-\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \sum_{i,j} \sum_{k,l=1}^2 [c_{k,l}^{i,j}]^2 \right] c_{n,m}^{i,j} \\ &\quad - \kappa v_{n,m}^{i,j}(t) \\ &\quad + (b_0 + b_1 \|u\|_{\mathbb{L}^2(\mathbb{D})} + \|\dot{u}\|_{\mathbb{L}^2(\mathbb{D})}) \alpha_{n,m}^{i,j} \dot{W}_{n,m}^{i,j}. \end{aligned} \quad (5)$$

subject to the initial conditions

$$(c_{n,m}^{i,j}(0), v_{n,m}^{i,j}(0)) = (c_{n,m,0}^{i,j}, v_{n,m,0}^{i,j})_{n,m \geq 1, i,j=1,2}$$

The solution of system (5) is exist and unique [8]. Now, we introduce the following standard definitions.

Definition 1: For $k \in \mathbb{N}$, take the partition $0 = t_0 < t_1 < t_2 < \dots < t_k = T$ of $[0, T]$ with current step sizes $h_k = t_{k+1} - t_k > 0$, then, as in Schurz [18], the **linear-implicit Euler-type method (LIEM)** is governed by the iterative scheme

$$\begin{aligned} c_{n,m}^{i,j}(t_{k+1}) &= c_{n,m}^{i,j}(t_k) + h_k v_{n,m}^{i,j}(t_{k+1}) \\ v_{n,m}^{i,j}(t_{k+1}) &= v_{n,m}^{i,j}(t_k) + h_k \left[f_{n,m}(u(t_k)) c_{n,m}^{i,j}(t_{k+1}) \right. \\ &\quad \left. - \kappa v_{n,m}^{i,j}(t_{k+1}) \right] + g_{n,m}(u(t_k)) \Delta_k W_{n,m}^{i,j} \end{aligned} \quad (6)$$

where

$$f_{n,m}(u(t_k)) = -\sigma^2(\lambda_n + \beta_m) + a_1 - a_2 \sum_{i,j=1}^2 \sum_{n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2$$

and

$$\begin{aligned} g_{n,m}(u(t_k)) &= (b_0 + b_1 \sqrt{\sum_{i,j=1}^2 \sum_{n,m=1}^N (c_{n,m}^{i,j}(t_k))^2} \\ &\quad + b_2 \sqrt{\sum_{i,j=1}^2 \sum_{n,m=1}^N (\dot{c}_{n,m}^{i,j}(t_k))^2}) \alpha_{n,m}^{i,j} \end{aligned}$$

and

$$\Delta_k W_{n,m}^{i,j} = W_{n,m}^{i,j}(t_{k+1}) - W_{n,m}^{i,j}(t_k) \in \mathcal{N}(0, h_k), h_k = t_{k+1} - t_k.$$

In [10], we proved that the explicit representation of LIEM as in the following theorem

Theorem 1: Assume that $|c_{n,m}^{i,j}(t_k)| < \infty$, $|v_{n,m}^{i,j}(t_k)| < \infty$, and $a_2 \geq 0$, $\kappa \geq 0$, and

$$\forall n, m \in \mathbb{N} : [-\sigma^2(\lambda_n + \beta_m) + a_1] h_k^2 < 1 + h_k \kappa.$$

Then the method (LIEM) governed by equation (6) and equation (7) has the non-exploding explicit representation

$$\begin{aligned} 1) c_{n,m}^{i,j}(t_{k+1}) &= \frac{(1 + h_k \kappa) c_{n,m}^{i,j}(t_k) + h_k v_{n,m}^{i,j}(t_k)}{1 + h_k \kappa - h_k^2 f_{n,m}(u(t_k))} \\ &\quad + \frac{h_k g_{n,m}(u(t_k)) \Delta_k W_{n,m}^{i,j}}{1 + h_k \kappa - h_k^2 f_{n,m}(u(t_k))} \end{aligned} \quad (8)$$

and

$$\begin{aligned} 2) v_{n,m}^{i,j}(t_{k+1}) &= \frac{v_{n,m}^{i,j}(t_k) + h_k f_{n,m}(u(t_k)) c_{n,m}^{i,j}(t_k)}{1 + h_k \kappa - h_k^2 f_{n,m}(u(t_k))} \\ &\quad + \frac{g_{n,m}(u(t_k)) \Delta_k W_{n,m}^{i,j}}{1 + h_k \kappa - h_k^2 f_{n,m}(u(t_k))} \end{aligned} \quad (9)$$

where $f_{n,m}(u(t_k))$, $g_{n,m}(u(t_k))$, $\Delta_k W_{n,m}^{i,j}$, and h_k as above.

Proof 2: (Theorem 1) See [10]

II. MEAN SQUARE CONSISTENCY OF NUMERICAL METHODS

Definition 2: A numerical method with the scheme

$$\hat{u}(t+h) = \hat{u}(t) + h f(\hat{u}(t)) + g(\hat{u}(t)) \Delta W(t) \quad (10)$$

where $h = t_{k+1} - t_k = \int_{t_k}^{t_{k+1}} ds$, $\Delta W^k = W(t_{k+1}) - W(t_k) = \int_{t_k}^{t_{k+1}} dW(s) \in \mathcal{N}(0, h)$ and $f(u(t))$ and $g(u(t))$ as above applied to SDE (5) is said to be **mean square consistent** with rate $r_2 > 0$ on \mathbb{D} iff $\exists K_2^c =$ consistency constant $\forall 0 \leq t \leq t+h \leq T$ where h is sufficiently small, i.e., $0 < h \leq \delta \leq 1$, $\forall u(t) \in H$, where $H := \{u \in \mathbb{L}^2(\mathbb{D}) | \dot{u} \in \mathbb{L}^2(\mathbb{D})\}$, and $\|u\|_H = \sqrt{\|u\|_{\mathbb{L}^2(\mathbb{D})}^2 + \|\dot{u}\|_{\mathbb{L}^2(\mathbb{D})}^2} : (\mathcal{F}_t, \mathcal{B}(\mathbb{L}^2))$ -measurable, then

$$\left(\mathbb{E} \left[\|u(t+h) - \hat{u}(t+h)\|_{\mathbb{N} \times \mathbb{N}}^2 | \mathcal{F}_t \right] \right)^{\frac{1}{2}} \leq K_2^c (V(u(t))) h^{r_2}$$

where

$$\hat{u}(t+h|t, u(t)) = \hat{u}(t) + \int_t^{t+h} f(u(s)) ds + \int_t^{t+h} g(u(s)) dW(s) ds$$

Lemma 3: (The Burkholder-Davis-Gundy Inequality)

$$\mathbb{E} \left\| \int_0^t g(s) dW(s) \right\|^p \leq C_B \mathbb{E} \left(\int_0^t \|g(s)\|^2 ds \right)^{\frac{p}{2}} \quad (11)$$

Proof 4: (Lemma 1) See Karatzas and Shreve [13].

Lemma 5: Let $X \sim \mathcal{N}(0, h_k)$, $\forall n \in \mathbb{N}$, $n \geq 1$, we have

$$\mathbb{E}(X)^{2n} = (2n - 1)!! h_k^n,$$

where $(2n - 1)!! = (2n - 1) \times (2n - 3) \times (2n - 5) \times \dots \times 5 \times 3 \times 1$.

Proof 6: (Lemma 2) See [9].

Theorem 7: The method (LIEM) given by

$$c_{n,m}^{i,j}(t_{k+1}) = c_{n,m}^{i,j}(t_k) + h_k v_{n,m}^{i,j}(t_{k+1}) \quad (12)$$

$$\begin{aligned} v_{n,m}^{i,j}(t_{k+1}) &= v_{n,m}^{i,j}(t_k) + h_k \left[f_{n,m}(u(t_k)) c_{n,m}^{i,j}(t_{k+1}) \right. \\ &\quad \left. - \kappa v_{n,m}^{i,j}(t_{k+1}) \right] + g_{n,m}(u(t_k)) \Delta_k W_{n,m}^{i,j} \end{aligned} \quad (13)$$

is locally mean square consistent with rate $r_2 \geq 1$, where $f(u(t_k))$ and $g(u(t_k))$ as above and

$$\Delta_k W_{n,m}^{i,j} = W_{n,m}^{i,j}(t_{k+1}) - W_{n,m}^{i,j}(t_k) \in \mathcal{N}(0, h_k), h_k = t_{k+1} - t_k.$$

Proof 8: (Theorem 2) We want to prove that

$$1) (\mathbb{E} [\|c(t+h) - \hat{c}(t+h)\|_N^2 |\hat{u}(t)])^{\frac{1}{2}} \leq K_{c_2}^c V(u(t)) h^{r_2} \quad (14)$$

$$2) (\mathbb{E} [\|v_{n,m}(t+h) - \hat{v}_{n,m}(t+h)\|_N^2 |\hat{u}(t)])^{\frac{1}{2}} \leq K_{v_2}^c V(u(t)) h^{r_2} \quad (15)$$

Provided that $c(t) = \hat{c}(t)$ and $v(t) = \hat{v}(t)$. To prove inequality (14), we know from [10] that

$$c(t+h) - \hat{c}(t+h) = \int_t^{t+h} (v(s) - \hat{f}_c(u(t))) ds - \int_t^{t+h} \hat{g}_c(u(t)) dW(s).$$

Thus,

$$\begin{aligned} & \|c_{n,m}(t+h) - \hat{c}_{n,m}(t+h)\|_N^2 \\ &= \left\| \int_t^{t+h} (v(s) - \hat{f}_c(u(t))) ds - \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \\ &\leq 2 \left\| \int_t^{t+h} (v(s) - \hat{f}_c(u(t))) ds \right\|_N^2 \\ &\quad + 2 \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2, \end{aligned}$$

by using the fact that $(a + b)^2 \leq 2a^2 + 2b^2$. So

$$\begin{aligned} \mathbb{E} \|c(t+h) - \hat{c}(t+h)\|_N^2 &\leq 2 \mathbb{E} \left\| \int_t^{t+h} (v(s) - \hat{f}_c(u(t))) ds \right\|_N^2 \\ &\quad + 2 \mathbb{E} \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \end{aligned}$$

which implies that

$$\begin{aligned} \mathbb{E} \|c(t+h) - \hat{c}(t+h)\|_N^2 &\leq 2 \int_t^{t+h} \mathbb{E} \|v(s) - \hat{f}_c(u(t))\|_N^2 ds \\ &\quad + 2 \mathbb{E} \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \quad (16) \end{aligned}$$

First, we will find the first part of the right hand side of inequality (16),

$$\begin{aligned} \|v(s) - \hat{f}_c(u(t))\|_N^2 &= \|v(s) - v(t) + v(t) - \hat{f}_c(u(t))\|_N^2 \\ &\leq 2 \|v(s) - v(t)\|_N^2 + 2 \|v(t) - \hat{f}_c(u(t))\|_N^2 \quad (17) \end{aligned}$$

Step 1: We know that

$$\begin{aligned} dv_{n,m}(r) &= [-\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \|u(r)\|_{\mathbb{L}^2(\mathbb{D})}^2] c_{n,m}(r) dr \\ &\quad - \kappa v_{n,m}(r) dr + g(u(r)) dW_{n,m}(r) \end{aligned}$$

thus

$$\begin{aligned} v_{n,m}(s) - v_{n,m}(t) &= \int_t^s [-\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \|u(r)\|_{\mathbb{L}^2(\mathbb{D})}^2] c_{n,m}(r) dr \\ &\quad - \kappa \int_t^s v_{n,m}(r) dr + \int_t^s g(u(r)) dW_{n,m}(r) \end{aligned}$$

so

$$\begin{aligned} & \|v_{n,m}(s) - v_{n,m}(t)\|_N^2 \\ &\leq 4 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \left\| \int_t^s c_{n,m}(r) dr \right\|_N^2 \\ &\quad + 4 a_2^2 \left\| \int_t^s \|u(r)\|_{\mathbb{L}^2(\mathbb{D})}^2 c_{n,m}(r) dr \right\|_N^2 \\ &\quad + 4 \kappa^2 \left\| \int_t^s v_{n,m}(r) dr \right\|_N^2 \\ &\quad + 4 \left\| \int_t^s g(u(r)) dW_{n,m}(r) \right\|_N^2 \end{aligned}$$

therefore,

$$\begin{aligned} & \mathbb{E} \|v_{n,m}(s) - v_{n,m}(t)\|_N^2 \\ &\leq 4 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} \left\| \int_t^s c_{n,m}(r) dr \right\|_N^2 \\ &\quad + 4 a_2^2 \mathbb{E} \left\| \int_t^s \|u(r)\|_{\mathbb{L}^2(\mathbb{D})}^2 c_{n,m}(r) dr \right\|_N^2 \\ &\quad + 4 \kappa^2 \mathbb{E} \left\| \int_t^s v_{n,m}(r) dr \right\|_N^2 \\ &\quad + 4 \mathbb{E} \left\| \int_t^s g(u(r)) dW_{n,m}(r) \right\|_N^2 \\ &\leq 4 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \int_s^t \mathbb{E} \|c(r)\|_N^2 dr \\ &\quad + 4 a_2^2 \int_s^t (\mathbb{E} \|u(r)\|_{\mathbb{L}^2(\mathbb{D})}^8)^{\frac{1}{2}} (\mathbb{E} \|c(r)\|_N^2)^{\frac{1}{2}} dr \\ &\quad + 4 \kappa^2 \int_s^t \mathbb{E} \|v(r)\|_N^2 dr \\ &\quad + 4 C_B \mathbb{E} \int_s^t \|g(u(r))\|_N^2 dW(r) \end{aligned}$$

by using Hölder and Burkholder-Davis-Gundy inequalities, Lemma 3, then substituting

$$g(u(t)) = (b_0 + b_1 \|u(t)\|_{l_2} + \|v(t)\|_{l_2}) \alpha^2,$$

we get

$$\begin{aligned} & \mathbb{E} \|v_{n,m}(s) - v_{n,m}(t)\|_N^2 \\ &\leq 4 \left([\sigma^2(\lambda_n + \beta_m) - a_1]^2 + \kappa^2 \right) \mathbb{E} V(u(s)) |s-t| \\ &\quad + 12 C_B \alpha^2 [b_0^2 + (b_1^2 + b_2^2) \mathbb{E} V(u(t))] |s-t| \\ &\quad + 4 a_2^2 \mathbb{E} V^2(u(t)) \mathbb{E} V(u(s)) |s-t| \\ &\leq \max_{1 \leq n, m \leq N} 4 \left([\sigma^2(\lambda_n + \beta_m) - a_1]^2 + \kappa^2 \right) \mathbb{E} V(u(s)) |s-t| \\ &\quad + 12 C_B \alpha^2 [b_0^2 + (b_1^2 + b_2^2) \mathbb{E} V(u(t))] |s-t| \\ &\quad + 8 a_2 \mathbb{E} V^2(u(t)) \mathbb{E} V(u(t)) |s-t|, \end{aligned}$$

and, if N is sufficiently large, we find

$$\begin{aligned} & \mathbb{E} \|v_{n,m}(s) - v_{n,m}(t)\|_N^2 \\ &\leq 4 \left(\left[\sigma^2 \pi^2 N^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right) \right]^2 + \kappa^2 \right) \mathbb{E} V(u(s)) |s-t| \\ &\quad + 12 C_B \alpha^2 [b_0^2 + (b_1^2 + b_2^2) \mathbb{E} V(u(t))] |s-t| \\ &\quad + 8 a_2 \mathbb{E} V^2(u(t)) \mathbb{E} V(u(t)) |s-t| \end{aligned}$$

taking the conditional expectation gives us

$$\begin{aligned} & \mathbb{E} \left(\|v_{n,m}(s) - v_{n,m}(t)\|_N^2 |u(t) \right) \\ & \leq 4 \left(\sigma^4 \pi^4 N^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + \kappa^2 + 3C_B \alpha^2 (b_1^2 + b_2^2) \right) V(u(t)) |s-t| \\ & + 12C_B \alpha^2 b_0^2 V(u(t)) |s-t| + 8a_2 V^2(u(t)) V(u(t)) |s-t| \\ & \leq K_5 (1 + V(u(t))) |s-t| \\ & \leq K_5 \tilde{V}(u(t)) |s-t| \end{aligned}$$

where

$$\begin{aligned} K_5 = & 4 \left(\sigma^4 \pi^4 N^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + \kappa^2 + 3C_B \alpha^2 (b_1^2 + b_2^2) \right) \\ & + 12C_B \alpha^2 b_0^2 + 8a_2 V^2(u(t)) \end{aligned}$$

therefore,

$$\begin{aligned} & \int_t^{t+h} \mathbb{E} [\|v_{n,m}(s) - v_{n,m}(t)\|_N^2 |u(t)] ds \\ & \leq k_5 \tilde{V}(u(t)) \frac{h_k^2}{2} \\ & \leq k_5 \tilde{V}(u(t)) h_k^2. \end{aligned} \tag{18}$$

Step 2: Also, we know that,

$$\begin{aligned} & v_{n,m}(t) - \hat{f}_c(u(t)) \\ & = \frac{(h_k \kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1]) v_{n,m}(t)}{1 + h_k \kappa + h_k^2 \left[\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2 \right]} \\ & + \frac{h_k^2 a_2 \left(\sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2 \right) v_{n,m}(t)}{1 + h_k \kappa + h_k^2 \left[\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2 \right]} \\ & + \frac{h_k [\sigma^2(\lambda_n + \beta_m) - a_1] c_{n,m}(t)}{1 + h_k \kappa + h_k^2 \left[\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2 \right]} \\ & + \frac{h_k a_2 \left(\sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2 \right) c_{n,m}(t)}{1 + h_k \kappa + h_k^2 \left[\sigma^2(\lambda_n + \beta_m) - a_1 + a_2 \sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2 \right]} \end{aligned}$$

to simplify, let

$$\begin{aligned} F(u(t)) = & 1 + h_k \kappa + h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1] \\ & + a_2 \sum_{i,j=1n,m=1}^2 \sum_{i,j=1n,m=1}^N [c_{n,m}^{i,j}(t_k)]^2, \end{aligned}$$

which gives $F(u(t)) \geq 0$. Now, if we pulling the expectation over

the squared norm of the latter identity, we find that

$$\begin{aligned} & \mathbb{E} \|v_{n,m}(t) - \hat{f}_c(u(t))\|_N^2 \\ & \leq 4 h_k^2 (\kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1])^2 \mathbb{E} \left\| \frac{v(t)}{F(u(t))} \right\|_N^2 \\ & + 4 h_k^4 a_2^2 \mathbb{E} \left\| \frac{\|u(t)\|_N^2 v(t)}{F(u(t))} \right\|_N^2 \\ & + 4 h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} \left\| \frac{c(t)}{F(u(t))} \right\|_N^2 \\ & + 4 h_k^2 a_2^2 \mathbb{E} \left\| \frac{\|u(t)\|_N^2 c(t)}{F(u(t))} \right\|_N^2 \end{aligned}$$

but we know that $F_c(u(t)) \geq 0$, so

$$\begin{aligned} & \mathbb{E} \|v(t) - \hat{f}_c(u(t))\|_N^2 \\ & \leq 4 h_k^2 (\kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1])^2 \mathbb{E} \|v(t)\|_N^2 \\ & + 4 h_k^4 a_2^2 \mathbb{E} \left\| \|u(t)\|_N^2 v(t) \right\|_N^2 \\ & + 4 h_k^2 (\sigma^2(\lambda_n + \beta_m) - a_1)^2 \mathbb{E} \|c(t)\|_N^2 \\ & + 4 a_2^2 h_k^2 \mathbb{E} \left\| \|u(t)\|_N^2 c(t) \right\|_N^2 \\ & \leq 4 h_k^2 (\kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1])^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^4 a_2^2 (\mathbb{E} \|u(t)\|_N^4)^{\frac{1}{2}} (\mathbb{E} \|v(t)\|_N^2)^{\frac{1}{2}} \\ & + 4 h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^2 a_2^2 (\mathbb{E} \|u(t)\|_N^4)^{\frac{1}{2}} (\mathbb{E} \|c(t)\|_N^2)^{\frac{1}{2}} \\ & \leq 4 h_k^2 (\kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1])^2 \\ & + h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^4 a_2^2 (\mathbb{E} V(u(t)))^{\frac{1}{2}} (\mathbb{E} V(u(t)))^{\frac{1}{2}} \\ & + 4 h_k^2 a_2^2 (\mathbb{E} V(u(t)))^{\frac{1}{2}} (\mathbb{E} V(u(t)))^{\frac{1}{2}} \end{aligned}$$

thus

$$\begin{aligned} & \mathbb{E} \|v(t) - \hat{f}_c(u(t))\|_N^2 \\ & \leq 4 h_k^2 (\kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1])^2 \mathbb{E} V(u(t)) \\ & + h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^2 a_2^2 (1 + h_k^2) \mathbb{E} V(u(t)) \\ & = 4 h_k^2 (\kappa + h_k [\sigma^2(\lambda_n + \beta_m) - a_1])^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^2 a_2^2 (1 + h_k^2) \mathbb{E} V(u(t)) \\ & \leq 4 h_k^2 (\kappa + \sigma^2(\lambda_n + \beta_m) - a_1)^2 \mathbb{E} V(u(t)) \\ & + 4 h_k^2 [\sigma^2(\lambda_n + \beta_m) - a_1]^2 \mathbb{E} V(u(t)) \\ & + 8 h_k^2 a_2^2 \mathbb{E} V(u(t)), \end{aligned}$$

for large N and take the conditional expectation, then we get

$$\begin{aligned} & \mathbb{E} \left[\|v(t) - \hat{f}_c(u(t))\|_N^2 |u(t) \right] \\ & \leq 4 \left[\kappa + \sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 \right] (1 + V(u(t))) h_k^2 \\ & + 4 \left[\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + 2 a_2^2 \right] (1 + V(u(t))) h_k^2 \end{aligned}$$

thus

$$\mathbb{E} \left[\|v(t) - \hat{f}_c(u(t))\|_N^2 |u(t) \right] \leq K_6 \tilde{V}(u(t)) h_k^2$$

where

$$K_6 = 4 \left[\kappa + \sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + \sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + 2a_2^2 \right].$$

Therefore,

$$\begin{aligned} & \int_t^{t+h} \mathbb{E} \left[\|v(t) - \hat{f}_c(u(t))\|_N^2 |u(t) \right] ds \\ & \leq K_6 \tilde{V}(u(t)) h_k^2 \int_t^{t+h} ds \\ & \leq K_6 \tilde{V}(u(t)) h_k^3 \end{aligned}$$

and since $h_k \leq 1$, then

$$\int_t^{t+h} \mathbb{E} \left[\|v(t) - \hat{f}_c(u(t))\|_N^2 |u(t) \right] ds \leq K_6 \tilde{V}(u(t)) h_k^2. \tag{19}$$

Form inequalities (20) and (21), we have

$$\begin{aligned} & \int_t^{t+h} \mathbb{E} \left[\|v(s) - \hat{f}_c(u(t))\|_N^2 |u(t) \right] ds \\ & \leq K_5 \tilde{V}(u(t)) h_k^2 + K_6 \tilde{V}(u(t)) h_k^2 \\ & \leq K_7 \tilde{V}(u(t)) h_k^2 \end{aligned} \tag{20}$$

where $K_7 = K_5 + K_6$. Now, the second part of inequality (16) is

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|^2 \\ & = \mathbb{E} \left\| \int_t^{t+h} \frac{h_k (b_0 + b_1 \|u(t)\|_{l_2} + b_2 \|v(t)\|_{l_2}) \alpha^2}{F_c(u(t))} dW(s) \right\|^2 \\ & = \mathbb{E} \left\| \frac{h_k (b_0 + b_1 \|u(t)\|_{l_2} + b_2 \|v(t)\|_{l_2}) \alpha^2}{F_c(u(t))} \int_t^{t+h} dW(s) \right\|_N^2. \end{aligned}$$

Then

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \\ & \leq \mathbb{E} \left[\left\| \frac{h_k (b_0 + b_1 \|u(t)\| + b_2 \|v(t)\|) \alpha^2}{F_c(u(t))} \right\|_N^2 \left\| \int_t^{t+h} dW(s) \right\|_N^2 \right] \\ & \leq \left(\mathbb{E} \left\| \frac{h_k (b_1 \|u(t)\| + b_2 \|v(t)\|) \alpha^2}{F_c(u(t))} \right\|_N^4 \right)^{\frac{1}{2}} \left(\mathbb{E} \left\| \int_t^{t+h} dW(s) \right\|_N^4 \right)^{\frac{1}{2}} \end{aligned}$$

using Lemma 5.1, $F_c(u(t)) \geq 0$, and

$$(a+b+c)^4 = ((a+b+c)^2)^2 \leq (3(a^2+b^2+c^2))^2 \leq 27(a^4+b^4+c^4),$$

then we find

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \\ & \leq h_k^2 \alpha^4 \left(27 \left[b_0^4 + b_1^4 \mathbb{E} \|u(t)\|^4 + b_2^4 \mathbb{E} \|v(t)\|^4 \right] \right)^{\frac{1}{2}} (3 h_k^2)^{\frac{1}{2}} \\ & \leq 9 \alpha^4 \left(b_0^4 + b_1^4 \mathbb{E} V(u(t)) + b_2^4 \mathbb{E} V^2(u(t)) \right)^{\frac{1}{2}} h_k^3. \end{aligned}$$

Hence, by redoing above steps for the conditional expectation, we arrive at

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \middle| \mathcal{F}_t \right] \\ & \leq 9 \alpha^4 \left(b_0^4 + b_1^4 V(u(t)) + b_2^4 V^2(u(t)) \right)^{\frac{1}{2}} h_k^3 \\ & \leq 9 \alpha^4 \left(b_0^2 + b_1^2 \sqrt{V(u(t))} + b_2^2 V(u(t)) \right) h_k^3 \\ & \leq 9 \alpha^4 \left(b_0^2 + \frac{b_1^4}{2} + \frac{V(u(t))}{2} + b_2^2 V(u(t)) \right) h_k^3, \end{aligned}$$

because we know from the Young's inequality that if $p = 2$ and $q = 2$, that $a \cdot b \leq \frac{a^2}{2} + \frac{b^2}{2}$, then

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \middle| \mathcal{F}_t \right] \\ & \leq 9 \alpha^4 \left(b_0^2 + \frac{b_1^4}{2} + \left(\frac{1}{2} + b_2^2 \right) V(u(t)) \right) h_k^3 \end{aligned}$$

and therefore

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 \middle| \mathcal{F}_t \right] \\ & \leq 9 \alpha^4 \max \left\{ b_0^2 + \frac{b_1^4}{2} + \frac{1}{2} + b_2^2, \frac{1}{2} + b_2^2 \right\} V(u(t)) h_k^2. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E} \left\| \int_t^{t+h} \hat{g}_c(u(t)) dW(s) \right\|_N^2 & \leq K_8 V(u(t)) h_k^2 \\ & \leq k_8 (1 + V(u(t))) h_k^2 \\ & \leq k_8 \tilde{V}(u(t)) h_k^2. \end{aligned} \tag{21}$$

where

$$K_8 = 9 \alpha^4 \max \left\{ b_0^2 + \frac{b_1^4}{2} + \frac{1}{2} + b_2^2, \frac{1}{2} + b_2^2 \right\}.$$

Thus inequality (16) equivalent to

$$\begin{aligned} & \mathbb{E} \left[\|c(t+h) - \hat{c}(t+h)\|_N^2 \middle| \mathcal{F}_t \right] \\ & \leq 2 \left(K_7 \tilde{V}(u(t)) h_k^2 + 2 K_8 \tilde{V}(u(t)) h_k^2 \right) \\ & \leq K_{c_2}^c \tilde{V}(u(t)) h_k^2 \end{aligned} \tag{22}$$

where $K_{c_2}^c = 2 (K_7 + K_8)$, which implies that

$$\left(\mathbb{E} \left[\|c(t+h) - \hat{c}(t+h)\|_N^2 \middle| \mathcal{F}_t \right] \right)^{\frac{1}{2}} \leq (K_{c_2}^c)^{\frac{1}{2}} \left(\tilde{V}(u(t)) \right)^{\frac{1}{2}} h_k. \tag{23}$$

2) To prove inequality (15), from [10] we know that,

$$\begin{aligned} v(t+h) - \hat{v}(t+h) & = \int_t^{t+h} [f_v(u(s)) - \hat{f}_v(u(t))] ds \\ & \quad + \int_t^{t+h} [g_v(u(s)) - \hat{g}_v(u(t))] dW(s) \end{aligned}$$

thus

$$\begin{aligned} & \|v(t+h) - \hat{v}(t+h)\|_N^2 \\ & \leq 2 \left\| \int_t^{t+h} [f_v(u(s)) - \hat{f}_v(u(t))] ds \right\|_N^2 \\ & \quad + 2 \left\| \int_t^{t+h} [g_v(u(s)) - \hat{g}_v(u(t))] dW(s) \right\|_N^2 \end{aligned} \tag{24}$$

and we know that

$$\begin{aligned} f_v(u(s)) - \hat{f}_v(u(t)) & = f_v(u(s)) - f_v(u(t)) + f_v(u(t)) \\ & \quad - f_v(\hat{u}(t)) + f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t)) \end{aligned}$$

and, by using the local property that $u(t) = \hat{u}(t)$, hence

$$\begin{aligned} \|f_v(u(s)) - \hat{f}_v(u(t))\|_N^2 & \leq 2 \|f_v(u(s)) - f_v(u(t))\|_N^2 \\ & \quad + 2 \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \end{aligned} \tag{25}$$

where

$$\begin{aligned} f_v(u(s)) - f_v(u(t)) & = [\sigma^2(\lambda_n + \beta_m) - a_1] (c(t) - c(s)) \\ & \quad + a_2 (\|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s)) \\ & \quad + \kappa (v(t) - v(s)) \end{aligned}$$

thus

$$\begin{aligned} \|f_v(u(s)) - f_v(u(t))\|_N^2 &\leq 3\left(\sigma^2(\lambda_n + \beta_m) - a_1\right)^2 \|c(t) - c(s)\|_N^2 \\ &\quad + 3a_2^2 \| \|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s) \|_N^2 \\ &\quad + 3\kappa^2 \|v(t) - v(s)\|_N^2. \end{aligned}$$

Also, from [10], we have

$$\begin{aligned} \| \|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s) \|_N \\ \leq 3 \|c(\gamma)\|_N^2 \|v(\gamma)\|_N \|c(t) - c(s)\|_N \end{aligned}$$

hence

$$\begin{aligned} \mathbb{E} \| \|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s) \|_N \\ \leq 9 \mathbb{E} \left(\|c(\gamma)\|_N^8 \|v(\gamma)\|_N^4 \|c(t) - c(s)\|_N^2 \right) \\ \leq 9 \left[\mathbb{E} \left(\|c(\gamma)\|_N^8 \|v(\gamma)\|_N^4 \right) \right]^{\frac{1}{2}} \left[\mathbb{E} \|c(t) - c(s)\|_N^4 \right]^{\frac{1}{2}} \\ \leq 9 \left[\mathbb{E} \|c(\gamma)\|_N^{16} \right]^{\frac{1}{2}} \left[\mathbb{E} \|v(\gamma)\|_N^8 \right]^{\frac{1}{2}} \left[\mathbb{E} \|c(t) - c(s)\|_N^4 \right]^{\frac{1}{2}} \\ \leq \frac{36}{a_2^2} \left[\mathbb{E} V^4(u(t)) \right]^{\frac{1}{2}} \left[\mathbb{E} V^4(u(t)) \right]^{\frac{1}{2}} \left[\mathbb{E} \|c(t) - c(s)\|_N^4 \right]^{\frac{1}{2}} \end{aligned}$$

therefore

$$\begin{aligned} \mathbb{E} \| \|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s) \|_N \leq \\ \frac{36}{a_2^2} \mathbb{E} V^4(u(t)) \left[\mathbb{E} \|c(t) - c(s)\|_N^4 \right]^{\frac{1}{2}} \end{aligned} \quad (26)$$

but we can find that

$$\begin{aligned} \mathbb{E} \|c(t) - c(s)\|_N^4 &\leq \mathbb{E} \left(\int_t^s [\mathbb{E} V(u(r))]^{\frac{1}{2}} dr \right)^4 \\ &\leq \int_t^s [\mathbb{E} V(u(r))]^2 dr (s-t)^2 \\ &\leq [\mathbb{E} V(u(t))]^2 (s-t)^2 \int_t^s dr \end{aligned}$$

and so

$$\mathbb{E} \|c(t) - c(s)\|_N^4 \leq [\mathbb{E} V(u(t))]^2 (s-t)^3. \quad (27)$$

Substituting inequality (27) in inequality (26), we find that

$$\begin{aligned} \mathbb{E} \| \|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s) \|_N \\ \leq \frac{36}{a_2^2} \mathbb{E} V^4(u(t)) [\mathbb{E} V(u(t))]^2 (s-t)^3 \end{aligned} \quad (28)$$

and we know that

$$\mathbb{E} \| \|c(t) - c(s)\|_N^2 c(s) \|_N \leq \mathbb{E} V(u(t)) (s-t)^2. \quad (29)$$

Also from Theorem 2 part 1), we know that

$$\int_t^{t+h} \mathbb{E} \|v(t) - v(s)\|_N^2 ds \leq K_5 \tilde{V}(u(t)) h_k^2. \quad (30)$$

Therefore, by using inequalities (28), (29), and (30), we find that

$$\begin{aligned} \int_t^{t+h} \mathbb{E} \|f_v(u(s)) - f_v(u(t))\|_N^2 ds \\ \leq 3 \left[\sigma^2(\lambda_n + \beta_m) - a_1 \right]^2 \int_t^{t+h} \mathbb{E} \|c(t) - c(s)\|_N^2 ds \\ + 3 a_2^2 \int_t^{t+h} \mathbb{E} \| \|u(t)\|_N^2 c(t) - \|u(s)\|_N^2 c(s) \|_N^2 ds \\ + 3 \kappa^2 \int_t^{t+h} \mathbb{E} \|v(t) - v(s)\|_N^2 ds, \end{aligned}$$

hence

$$\begin{aligned} \int_t^{t+h} \mathbb{E} [\|f_v(u(s)) - f_v(u(t))\|_N^2 | u(t)] ds \\ \leq 3 \left[\sigma^2(\lambda_n + \beta_m) - a_1 \right]^2 \int_t^{t+h} V(u(t))(s-t)^2 ds \\ + 108 \int_t^{t+h} V^4(u(t)) \mathbb{E} V(u(t)) |s-t|^3 ds \\ + 3 K_5 \kappa^2 \tilde{V}(u(t)) h_k^2 \\ \leq \left[\sigma^2(\lambda_n + \beta_m) - a_1 \right]^2 V(u(t)) h_k^3 \\ + \frac{108}{4} V^4(u(t)) V(u(t)) h_k^4 + 3 K_5 \kappa \tilde{V}(u(t)) h_k^2 \\ \leq \left[\sigma^2(\lambda_n + \beta_m) - a_1 \right]^2 V(u(t)) h_k^2 \\ + 27 V^4(u(t)) V(u(t)) h_k^2 + 3 K_5 \kappa \tilde{V}(u(t)) h_k^2. \end{aligned}$$

Thus, for large N , we have

$$\begin{aligned} \int_t^{t+h} \mathbb{E} [\|f_v(u(s)) - f_v(u(t))\|_N^2 | u(t)] ds \\ \leq \sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 V(u(t)) h_k^2 \\ + 27 V^4(u(t)) V(u(t)) h_k^2 + 3 K_5 \kappa \tilde{V}(u(t)) h_k^2 \\ \leq \sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 (1 + V(u(t))) h_k^2 \\ + 27 V^4(u(t)) (1 + V(u(t))) h_k^2 + 3 K_5 \kappa \tilde{V}(u(t)) h_k^2 \end{aligned}$$

therefore

$$\int_t^{t+h} \mathbb{E} [\|f_v(u(s)) - f_v(u(t))\|_N^2 | u(t)] ds \leq K_9 \tilde{V}(u(t)) h_k^2 \quad (31)$$

where

$$K_9 = \sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + 27 V^4(u(t)) + 3 K_5 \kappa.$$

The second part of inequality (25) is,

$$\begin{aligned} & \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq h_k^2 \left[\left\| \frac{\kappa \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right] \hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \\ & \quad + \left\| \frac{h_k \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 + \left\| \frac{\kappa^2 \hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \\ & \quad \left. + \left\| \frac{h_k \kappa \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right] \hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right] \\ & \leq 4 h_k^2 \left[\kappa^2 \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \\ & \quad + h_k^2 \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^4 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \\ & \quad + \kappa^4 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \\ & \quad + h_k^2 \kappa^2 \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \left. \right] \\ & \leq 4 h_k^2 \left[\kappa^2 \left(\left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \right. \\ & \quad \left. \left. + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right) \right. \\ & \quad \left. + h_k^2 \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left(\left[\sigma^2 (\lambda_n + \beta_m) - a_1 \right. \right. \right. \\ & \quad \left. \left. \left. + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right) \right] \end{aligned}$$

hence

$$\begin{aligned} & \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq 4 h_k^2 \left[\left(\kappa^2 + h_k^2 \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right) \right. \\ & \quad \left(\left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \right. \\ & \quad \left. \left. + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right) \right] \end{aligned}$$

using $F(\hat{u}(t)) \geq 1$, and pulling the expectation over the last identity, then we find

$$\begin{aligned} & \mathbb{E} \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq 4 h_k^2 \mathbb{E} \left[\left(\kappa^2 + h_k^2 \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right) \left(\left[\sigma^2 (\lambda_n + \beta_m) \right. \right. \right. \right. \\ & \quad \left. \left. \left. - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right) \right] \end{aligned}$$

therefore,

$$\begin{aligned} & \mathbb{E} \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq 4 h_k^2 \mathbb{E} \left[\left(\kappa^2 + \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right) \right. \\ & \quad \left(\left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \right. \\ & \quad \left. \left. + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right) \right] \end{aligned}$$

hence

$$\begin{aligned} & \mathbb{E} \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq 4 h_k^2 \left(\mathbb{E} \left[\kappa^2 + \left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right]^2 \right)^{\frac{1}{2}} \\ & \quad \left(\mathbb{E} \left[\left[\sigma^2 (\lambda_n + \beta_m) - a_1 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right]^2 \right)^{\frac{1}{2}} \end{aligned}$$

for large N , we have

$$\begin{aligned} & \mathbb{E} \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq 4 h_k^2 \left(\mathbb{E} \left[\kappa^2 + \left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right]^2 \right)^{\frac{1}{2}} \\ & \quad \left(\mathbb{E} \left[\left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right]^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{32}$$

First, I will calculate the first part of the right side of inequality (32) which is

$$\begin{aligned} & \left(\mathbb{E} \left[\kappa^2 + h_k^2 \left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right]^2 \right)^{\frac{1}{2}} \\ & \leq \left(\mathbb{E} \left[2 \kappa^4 + 2 \left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^4 \right] \right)^{\frac{1}{2}} \\ & \leq \left(\mathbb{E} \left[2 \kappa^4 + 2 \left[4 \left(\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 \right)^4 + 4 a_2^4 \|\hat{u}(t)\|_N^8 \right] \right] \right)^{\frac{1}{2}} \end{aligned}$$

hence

$$\begin{aligned} & \left(\mathbb{E} \left[\kappa^2 + h_k^2 \left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \right]^2 \right)^{\frac{1}{2}} \\ & \leq \left(2 \kappa^4 + 2 \left[4 \left(\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 \right)^4 + 4 a_2^4 \mathbb{E} \|\hat{u}(t)\|_N^8 \right] \right)^{\frac{1}{2}} \\ & \leq \left(2 \kappa^4 + 8 \left[\left(\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 \right)^4 + a_2^4 \mathbb{E} V^2(u(t)) \right] \right)^{\frac{1}{2}} \\ & = \left(2 \kappa^4 + 8 \left(\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 \right)^4 + 8 a_2^4 \mathbb{E} V^2(u(t)) \right)^{\frac{1}{2}} \\ & \leq \sqrt{2} \left(\kappa^2 + 2 \left(\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 \right)^2 + 2 a_2^2 [\mathbb{E} V^2(u(t))] \right)^{\frac{1}{2}} \\ & \leq \sqrt{2} \left(\kappa^2 + 2 \sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2 a_2^2 [\mathbb{E} V^2(u(t))] \right)^{\frac{1}{2}}. \end{aligned}$$

Now, the second part of the right side of inequality (32) is

$$\begin{aligned} & \mathbb{E} \left[\left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^2 + \kappa^2 \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \right]^2 \\ & \leq 2 \mathbb{E} \left(\left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^4 \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^4 \right) \\ & \quad + 2 \kappa^2 \mathbb{E} \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \\ & \leq 2 \left(\mathbb{E} \left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^8 \right)^{\frac{1}{2}} \left(\mathbb{E} \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^8 \right)^{\frac{1}{2}} \\ & \quad + 2 \kappa^2 \mathbb{E} \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \\ & \leq 2 \left(16 \sigma^{16} N^{16} \pi^{16} \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^{16} + a_2^8 \mathbb{E} \|\hat{u}(t)\|_N^{16} \right)^{\frac{1}{2}} \\ & \quad \left(\mathbb{E} \left\| \frac{\hat{c}(t)}{F(\hat{u}(t))} \right\|_N^8 \right)^{\frac{1}{2}} + 2 \kappa^2 \mathbb{E} \left\| \frac{\hat{v}(t)}{F(\hat{u}(t))} \right\|_N^2 \end{aligned}$$

but $F(\hat{u}(t)) \geq 1$, then

$$\begin{aligned} & \mathbb{E} \left[\left(\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right)^2 \|\hat{c}(t)\|_N^2 + \kappa^2 \|\hat{v}(t)\|_N^2 \right)^2 \\ & \leq 2 \left(16\sigma^{16} N^{16} \pi^{16} \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^{16} + a_2^8 \mathbb{E} \|\hat{u}(t)\|_N^{16} \right)^{\frac{1}{2}} (\mathbb{E} \|\hat{c}(t)\|_N^8)^{\frac{1}{2}} \\ & \quad + 2\kappa^2 \mathbb{E} \|\hat{v}(t)\|_N^2 \\ & \leq 2 \left(4\sigma^8 N^8 \pi^8 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^8 + a_2^4 \mathbb{E} \|\hat{u}(t)\|_N^8 \right) (\mathbb{E} \|\hat{c}(t)\|_N^8)^{\frac{1}{2}} \\ & \quad + 2\kappa^2 \mathbb{E} \|\hat{v}(t)\|_N^2, \end{aligned}$$

hence

$$\begin{aligned} & \left[\mathbb{E} \left(\left[\sigma^2 N^2 \pi^2 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^2 + a_2 \|\hat{u}(t)\|_N^2 \right]^2 \|\hat{c}(t)\|_N^2 + \kappa^2 \|\hat{v}(t)\|_N^2 \right)^2 \right]^{\frac{1}{2}} \text{ hence} \\ & \leq \left[2 \left(4\sigma^8 N^8 \pi^8 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^8 + a_2^4 \mathbb{E} \|\hat{u}(t)\|_N^8 \right) (\mathbb{E} \|\hat{c}(t)\|_N^8)^{\frac{1}{2}} + 2\kappa^2 \mathbb{E} \|\hat{v}(t)\|_N^2 \right]^{\frac{1}{2}} \\ & \leq \sqrt{2} \left(4\sigma^8 N^8 \pi^8 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^8 + a_2^4 \mathbb{E} \|\hat{u}(t)\|_N^8 \right)^{\frac{1}{2}} (\mathbb{E} \|\hat{c}(t)\|_N^8)^{\frac{1}{4}} \\ & \quad + \sqrt{2} \kappa (\mathbb{E} \|\hat{v}(t)\|_N^2)^{\frac{1}{2}} \\ & \leq \sqrt{2} \left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + a_2^2 (\mathbb{E} \|\hat{u}(t)\|_N^8)^{\frac{1}{2}} \right) (\mathbb{E} \|\hat{c}(t)\|_N^8)^{\frac{1}{4}} \\ & \quad + \sqrt{2} \kappa (\mathbb{E} \|\hat{v}(t)\|_N^2)^{\frac{1}{2}} \\ & \leq \sqrt{2} \left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2 (\mathbb{E} V^2(u(t)))^{\frac{1}{2}} \right) \\ & \quad \left(\frac{2}{a_2} \mathbb{E} V^8(u(t)) \right)^{\frac{1}{4}} + \sqrt{2} \kappa (\mathbb{E} V(u(t)))^{\frac{1}{2}}. \end{aligned}$$

Thus inequality (32) equivalent to

$$\begin{aligned} & \mathbb{E} \|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 \\ & \leq 4h_k^2 \cdot \sqrt{2} \left(\kappa^2 + 2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2^2 [\mathbb{E} V^2(u(t))]^{\frac{1}{2}} \right) \\ & \quad \left[\sqrt{2} \left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2 (\mathbb{E} V^2(u(t)))^{\frac{1}{2}} \right) \right. \\ & \quad \left. \left(\frac{2}{a_2} \mathbb{E} V^8(u(t)) \right)^{\frac{1}{4}} + \sqrt{2} \kappa (\mathbb{E} V(u(t)))^{\frac{1}{2}} \right]. \end{aligned}$$

Pulling the conditional expectation over the last identity, we find that

$$\begin{aligned} & \mathbb{E} \left[\|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 | u(t) \right] \\ & \leq 8h_k^2 \left(\kappa^2 + 2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2^2 [V^2(u(t))]^{\frac{1}{2}} \right) \\ & \quad \left[\left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2 (V^2(u(t)))^{\frac{1}{2}} \right) \right. \\ & \quad \left. \left(\frac{2}{a_2} V^8(u(t)) \right)^{\frac{1}{4}} + \kappa (V(u(t)))^{\frac{1}{2}} \right]. \end{aligned}$$

Thus

$$\begin{aligned} & \mathbb{E} \left[\|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 | u(t) \right] \\ & \leq 8h_k^2 \left(\kappa^2 + 2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2^2 V(u(t)) \right) \\ & \quad \left[\left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2 V(u(t)) \right) \right. \\ & \quad \left. \left(\frac{2}{a_2} \right)^{\frac{1}{4}} V^2(u(t)) + \kappa V^{\frac{1}{2}}(u(t)) \right] \\ & \leq 8h_k^2 \cdot \left(\kappa^2 + 2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2^2 V(u(t)) \right) \\ & \quad \left[\left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2 V(u(t)) \right) \right. \\ & \quad \left. \left(\frac{2}{a_2} \right)^{\frac{1}{4}} V^{\frac{3}{2}}(u(t)) + \kappa \right] V^{\frac{1}{2}}(u(t)) \end{aligned}$$

$$\begin{aligned} & \mathbb{E} \left[\|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 | u(t) \right] \\ & \leq K_{10} (1 + V(u(t))) h_k^2 \\ & \leq K_{10} \tilde{V}(u(t)) h_k^2. \end{aligned} \tag{33}$$

where

$$\begin{aligned} K_{10} &= 8 \cdot \left(\kappa^2 + 2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2^2 V(u(t)) \right) \\ & \quad \left[\left(2\sigma^4 N^4 \pi^4 \left(\frac{1}{l_x^2} + \frac{1}{l_y^2} \right)^4 + 2a_2 V(u(t)) \right) \left(\frac{2}{a_2} \right)^{\frac{1}{4}} V^{\frac{3}{2}}(u(t)) + \kappa \right], \end{aligned}$$

therefore

$$\begin{aligned} & \int_t^{t+h} \mathbb{E} \left[\|f_v(\hat{u}(t)) - \hat{f}_v(\hat{u}(t))\|_N^2 | u(t) \right] ds \\ & \leq K_{10} \tilde{V}(u(t)) h_k^3 \\ & \leq K_{10} \tilde{V}(u(t)) h_k^2. \end{aligned} \tag{34}$$

To complete the proof, we have to simplify the second part of the right side of inequality (24) which is

$$\begin{aligned} & \left\| \int_t^{t+h} \left[g_v(u(s)) - \hat{g}_v(u(t)) \right] dW(s) \right\|_N^2 \\ & \leq C_B \int_t^{t+h} \|g_v(u(s)) - \hat{g}_v(u(t))\|_N^2 ds \end{aligned}$$

but from [10] we know that

$$\begin{aligned} & \left\| \int_t^{t+h} \left[g_v(u(s)) - \hat{g}_v(u(t)) \right] dW(s) \right\|_N^2 \\ & \leq C_B \int_t^{t+h} \|g_v(u(s)) - g_v(u(t))\|_N^2 ds \\ & \quad + C_B \int_t^{t+h} \|g_v(u(t)) - \hat{g}_v(u(t))\|_N^2 ds. \end{aligned} \tag{35}$$

We start with the first part of the right side of inequality (35) which is

$$\begin{aligned} & \int_t^{t+h} \|g_v(u(s)) - g_v(u(t))\|_N^2 ds \\ & \leq \int_t^{t+h} \|b_1 (\|u(s)\|_{l^2} - \|u(t)\|_{l^2}) + b_2 (\|v(s)\|_{l^2} - \|v(t)\|_{l^2})\|_N^2 ds \\ & \leq \int_t^{t+h} [2b_1^2 (\|u(s)\|_{l^2} - \|u(t)\|_{l^2})^2 + 2b_2^2 (\|v(s)\|_{l^2} - \|v(t)\|_{l^2})^2] ds \\ & \leq \int_t^{t+h} \max\{2b_1^2, 2b_2^2\} [(\|u(s)\|_{l^2} - \|u(t)\|_{l^2})^2 + (\|v(s)\|_{l^2} - \|v(t)\|_{l^2})^2] ds \\ & \leq \int_t^{t+h} 2 \max\{2b_1^2, 2b_2^2\} [\|u(\gamma)\|_{l^2}^2 |s-t| + \|v(\gamma)\|_{l^2}^2 |s-t|] ds \\ & \leq \max\{2b_1^2, 2b_2^2\} V(u(t)) h_k^2 \end{aligned}$$

thus

$$\begin{aligned} \int_t^{t+h} \mathbb{E} \|g_v(u(s)) - g_v(u(t))\|_N^2 ds \\ \leq K_{11} V(u(t)) h_k^2 \leq K_{11} \tilde{V}(u(t)) h_k^2 \end{aligned} \quad (36)$$

where $K_{11} = \max\{2b_1^2, 2b_2^2\}$. Also, the second part of the right side of inequality (35), we start with

$$\begin{aligned} & \mathbb{E} [\|g_v(u(t)) - \hat{g}_v(u(t))\|_N^2] \\ & = h_k \mathbb{E} \left[\left\| \frac{[\kappa + h_k F(u(t))] [b_0 + b_1 \|u(t)\|_{l^2} + b_2 \|v(t)\|_{l^2}]}{1 + h_k \kappa + h_k^2 F(u(t))} \right\|_N^2 \right] \\ & \leq h_k (\mathbb{E} \|\kappa + h_k F(u(t))\|_N^4)^{\frac{1}{2}} (\mathbb{E} \|b_0 + b_1 \|u(t)\|_{l^2} + b_2 \|v(t)\|_{l^2}\|_N^4)^{\frac{1}{2}} \\ & \leq 81 h_k (\mathbb{E} [\kappa^4 + h_k^4 \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + a_2^4 \|u(t)\|_{l^2}^8])^{\frac{1}{2}} \\ & \quad \cdot (\mathbb{E} [b_0^4 + b_1^4 \|u(t)\|_{l^2}^4 + b_2^4 \|v(t)\|_{l^2}^4])^{\frac{1}{2}}, \end{aligned}$$

hence

$$\begin{aligned} & \mathbb{E} [\|g_v(u(t)) - \hat{g}_v(u(t))\|_N^2 |u(t)] \\ & \leq 81 h_k \left(\kappa^4 + h_k^4 \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + a_2^4 \|u(t)\|_{l^2}^8 \right)^{\frac{1}{2}} \\ & \quad \cdot (b_0^4 + b_1^4 \|u(t)\|_{l^2}^4 + b_2^4 \|v(t)\|_{l^2}^4)^{\frac{1}{2}} \\ & \leq 81 h_k \left(\kappa^4 + h_k^4 \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + 4a_2^2 \mathbb{E} V^2(u(t)) |u(t) \right)^{\frac{1}{2}} \\ & \quad \cdot \left(b_0^4 + \frac{2b_1^4}{a_2} \mathbb{E} V(u(t)) |u(t) + b_2^4 \mathbb{E} V^2(u(t)) |u(t) \right)^{\frac{1}{2}} \\ & \leq 81 h_k \left(\kappa^4 + h_k^4 \sigma^4 N^4 \pi^4 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^4 + 4a_2^2 V^2(u(t)) \right)^{\frac{1}{2}} \\ & \quad \cdot \left(b_0^4 + \frac{2b_1^4}{a_2} V(u(t)) + b_2^4 V^2(u(t)) \right)^{\frac{1}{2}} \\ & \leq 81 h_k \left(\kappa^2 + h_k^2 \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2a_2 V(u(t)) \right) \\ & \quad \cdot \left(b_0^2 + \frac{\sqrt{2} b_1^2}{\sqrt{a_2}} (V(u(t)))^{\frac{1}{2}} + b_2^2 V(u(t)) \right) \\ & \leq 81 \left(\kappa^2 + h_k^2 \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2a_2 (V(u(t)))^{\frac{1}{2}} \right) \\ & \quad \cdot \left(b_0^2 + \frac{\sqrt{2} b_1^2}{\sqrt{a_2}} + b_2^2 (V(u(t)))^{\frac{1}{2}} \right) (V(u(t)))^{\frac{1}{2}} h_k \\ & \leq 81 \left(\kappa^2 + h_k^2 \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2a_2 (V(u(t)))^{\frac{1}{2}} \right) \\ & \quad \cdot \left(b_0^2 + \frac{\sqrt{2} b_1^2}{\sqrt{a_2}} + b_2^2 (V(u(t)))^{\frac{1}{2}} \right) (1 + V(u(t))) h_k \end{aligned}$$

thus

$$\mathbb{E} [\|g_v(u(t)) - \hat{g}_v(u(t))\|_N^2 |u(t)] \leq K_{12} \tilde{V}(u(t)) h_k \quad (37)$$

where

$$\begin{aligned} K_{12} & = 81 \left(\kappa^2 + h_k^2 \sigma^2 N^2 \pi^2 (\frac{1}{l_x^2} + \frac{1}{l_y^2})^2 + 2a_2 (V(u(t)))^{\frac{1}{2}} \right) \\ & \quad \cdot \left(b_0^2 + \frac{\sqrt{2} b_1^2}{\sqrt{a_2}} + b_2^2 (V(u(t)))^{\frac{1}{2}} \right) \end{aligned}$$

therefore

$$\int_t^{t+h} \mathbb{E} [\|g_v(u(t)) - \hat{g}_v(u(t))\|_N^2 |u(t)] ds \leq K_{12} \tilde{V}(u(t)) h_k^2. \quad (38)$$

Thus by using inequalities (31), (34), (36), and (38) we can write inequality (15) as

$$\begin{aligned} & \mathbb{E} [\|v_{n,m}(t+h) - \hat{v}_{n,m}(t+h)\|_N^2 | \hat{u}(t)] \\ & \leq [4(K_9 + K_{10}) + 2C_B(K_{11} + K_{12})] \tilde{V}(u(t)) h_k^2 \\ & \leq K_{v_2}^c \tilde{V}(u(t)) h_k^2 \end{aligned} \quad (39)$$

where $K_{v_2}^c = 4(K_9 + K_{10}) + 2C_B(K_{11} + K_{12})$. Thus

$$\begin{aligned} & \left(\mathbb{E} [\|v_{n,m}(t+h) - \hat{v}_{n,m}(t+h)\|_N^2 | \hat{u}(t)] \right)^{\frac{1}{2}} \leq K_{v_2}^c \left(\tilde{V}(u(t)) \right)^{\frac{1}{2}} h_k \\ & \leq K_{v_2}^c \tilde{V}(u(t)) h_k. \end{aligned} \quad (40)$$

III. CONCLUSION

In this article we prove that the linear-implicit Euler method is mean square consistent with rate $r_0 = 2$ on nonlinear stochastic wave equations in 2 dimensions. In the future, we will continue to show that the LIEM is mean square and asymptotic stability, and convergence.

Also, we will study the stability and mean square consistency of the solution of stochastic heat equation, Burger equation.

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