

On a Generalized Squared Gaussian Diffusion Model for Option Valuation

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Abstract. In financial mathematics, option pricing models are vital tools whose usefulness cannot be overemphasized. Modern approaches and modelling of financial derivatives are therefore required in option pricing and valuation settings. In this paper, we derive via the application of Ito lemma, a pricing model referred to as Generalized Squared Gaussian Diffusion Model (GSGDM) for option pricing and valuation. Same approach can be considered via Stratonovich stochastic dynamics. We also show that the classical Black-Scholes, and the square root constant elasticity of variance models are special cases of the GSGDM. In addition, general solution of the GSGDM is obtained using modified variational iterative method (MVIM).

1 Introduction

In modern finance, the importance of options in pricing theory cannot be overemphasized as they can be used for control risk and hedging. This therefore requires the attention of financial analysts when dealing with finance, actuarial sciences, and other related areas of applied sciences [1-3]. Hence, the involvement of stock options in the study of option pricing and valuation theory [4]. Black and Scholes [5, 6] assumed the underlying stock price process, S_t , to be lognormally distributed, and as such solves the following stochastic dynamics (SDE):

$$dS_t = S_t (\mu dt + \sigma dW_t) \quad (1.1)$$

where μ and σ are mean rate of return, and stock price volatility parameter respectively, while W_t represents a one-dimensional Standard Brownian Motion (SBM).

According to the Black-Scholes model, the value of a European-style option on S stock at time t , is $V = V(S, t)$ that solves the partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (1.2)$$

Equation (1.2) is the classical Black-Scholes model, where r is the risk-free rate, $V \in C^{2,1}(\mathbb{R} \times [0, T])$, with a payoff function $p_f(S, t)$, and expiration price, E such that:

$$p_f(S, t) = \begin{cases} (S - E)^+, & \text{for European call option} \\ (E - S)^+, & \text{for European put option} \end{cases} \quad (1.3)$$

Black and Scholes based their model on some assumptions such as arbitrage-free opportunities, no allowance of dividend-yield paying stock, lack of transaction costs, constant mean rate and constant volatility parameter among others [7]. To be addressed here among the assumptions is that of constant volatility, that is the stock price standard deviation. The constant nature of the volatility as assumed by Black and Scholes was to make the model a linear type in order to obtain analytical solution easily. Contrary to this, constant volatility appears unrealistic in practical settings. We therefore intend to resort to a more accommodating model driven by constant elasticity of variance (CEV).

2 Basic Definitions, Theorem(s) and Existing Model

The assumption of constant volatility as in (1.1) has drawn the attention of many researchers; see [8-10] and the references therein for more details. Delbean and Shirakawa [11] proved that the CEV model permits arbitrage opportunities when the stock price is based on strict positive conditions.

Cox and Ross [12] considered the CEV diffusion process governed by the SDE:

$$dS_t = S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} d(W_t) \quad (2.1)$$

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whose solution is S_t , ξ represents an elasticity rate, while μ , σ , and W_t remain as earlier defined.

Beckners [13] considered the CEV and its implications for option pricing on the basis of empirical studies and concluded that the CEV class could be a better descriptor of the actual stock price in terms of behaviour than the traditionally used lognormal model. MacBeth and Merville [14] proposed a three-stage-procedure on how to estimate σ and ξ .

In what follows, the Black-Scholes model will be modified using the SDE associated with the CEV model in (2.1) to get the the GSGDM as a proposed model.

Definition 2a: Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable with $E(X)$ and $Var(X)$ denoting the mathematical expectation and the variance of X respectively. Then we denote by $SG_{\Xi}(\Omega)$, the space of all square-Gaussian random variables.

Definition 2b: [15] A random process

$X = (X(t), t \in T)$ is said to be square-Gaussian, if for all $t \in T$ $X(t) \in SG_{\Xi}(\Omega)$ and

$$\sup_{t \in T} \left(E(X(t))^2 \right)^{\frac{1}{2}} < \infty .$$

Note: In this work, we take all stochastic processes to be square Gaussian (SG).

Theorem 2.1 : [16]

Suppose $X(t)$ an adapted stochastic process on a filtered probability space $F_{PS} = (\Omega, B, \mu, F(B))$, possesses a quadratic variation $\langle X \rangle$, is a solution of the SDE:

$$dX(t) = H_1 dt + H_2 dW(t) \tag{2.2}$$

where $H_1 = H_1(t, X(t))$ represents the drift coefficient, $H_2 = H_2(t, X(t))$ is the diffusion coefficient, $W(t)$ is a standard Brownian motion representing the intrinsic noise (white noise) in the dynamical system; and $u \in C^{1,2}(T \times \mathbb{R})$, such that $u(t, x) : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a twice continuously differentiable function with T as time at maturity, then, $u = u(t, X(t))$ is a stochastic process for which:

$$du(t, X(t)) = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX(t) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dX(t))^2 \tag{2.3}$$

whence,

$$\begin{cases} du(t, X(t)) = \left(\frac{\partial u}{\partial t} + H_1 \frac{\partial u}{\partial x} + \frac{1}{2} H_2^2 \frac{\partial^2 u}{\partial x^2} \right) dt \\ + H_2 \frac{\partial u}{\partial x} dW(t) \end{cases} \tag{2.4}$$

Equation (2.4) is called an Ito formula.

Note: If $X = W$, that is $H_1 \equiv 0$ and $H_2 \equiv 1$, then

$$du(t, X(t)) = \left(\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \right) dt + \frac{\partial u}{\partial x} dW(t). \tag{2.5}$$

Equation (2.5) is called an Ito lemma.

3 VIM and the Modified VIM [17, 18]

Considering the general nonlinear PDE of the form:

$$\begin{cases} Lu(x, t) + Ru(x, t) + Nu(x, t) = f(x, t) \\ u(x, t) = h(x) \end{cases} \tag{3.1}$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator whose partial derivatives are w.r.t. x , $Nu(x, t)$ is a nonlinear term associated to (3.1) and $f(x, t)$ is a source term (which may be homogeneous or inhomogeneous), thus by the classical VIM, the solution of (3.1) is expressed as:

$$\begin{cases} u_{n+1}(x, t) = u_n(x, t) \\ + \int_0^t \lambda (Lu_n + Ru_n + Nu_n - f(x, t)) ds \end{cases} \tag{3.2}$$

where λ is a Lagrange multiplier [19, 20] to be identified optimally via variational theory, and the terms: Ru_n and Nu_n are being considered as restricted variations such that $\delta Ru_n = 0$ and $\delta Nu_n = 0$. Hence, by calculating the variations w.r.t. u_n using the stationary conditions:

$$\begin{cases} \lambda'(x) = 0 \\ 1 + \lambda(x) = 0, \end{cases} \tag{3.3}$$

the Lagrange multiplier is identified as $\lambda = -1$. Therefore, (3.2) becomes:

$$\begin{cases} u_{n+1}(x, t) = u_n(x, t) \\ - \int_0^t (Lu_n + R(u_n) + N(u_n) - f(x, t)) ds. \end{cases} \tag{3.4}$$

Remark: Using (3.4) for the solution of special kind of nonlinear differential equations involves the calculation of unrequired terms, repeated calculations, and time-consumption, hence, the need for meaningful modification of the VIM. The modified VIM as proposed by [17] gives the iterative formula as follows:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (R(u_n - u_{n-1}) + (G_n - G_{n-1})) ds, \tag{3.5}$$

where

$$\begin{cases} u_{-1} = 0, \\ u_0 = h(x), \\ u_1 = u_0 - \int_0^t (R(u_n - u_{n-1}) + (G_n - G_{n-1}) - f) ds, \\ G_n = Nu_n + O(t^{n+1}). \end{cases} \tag{3.6}$$

3.1 Derivation of the Generalized Model (GSGDM)

Suppose the stock price, S_t at time t , satisfies the SDE in (2.1), with all parameters as earlier defined, and that the value of the contingent claim $V = V(S, t)$ is such that $V \in C^{2,1}(\mathbb{R} \times [0, T])$, therefore by Ito lemma, we have:

$$\begin{cases} dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \langle dS \rangle \\ = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} \left(S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} dW_t \right) \\ + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left\langle S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} dW_t \right\rangle \end{cases} \tag{3.7}$$

Thus, for:

$$\left\langle S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} d(W_t) \right\rangle = \sigma^2 S_t^{\xi} dt, \tag{3.8}$$

we therefore write (3.7) as:

$$dV = \left(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^{\xi} \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S^{\frac{\xi}{2}} \frac{\partial V}{\partial S} dW. \tag{3.9}$$

Let $\Phi(t)$ be a delta-hedged-portfolio by longing a contingent, and shorting a delta unit of the underlying asset such that:

$$\begin{aligned} \Phi(t) &= V(S, t) - \Delta S, \quad d\Phi(t) = dV(s, t) - \Delta dS, \\ \Delta &= \frac{\partial V}{\partial S}, \text{ and } d\Phi(t) = r\Phi(t)dt \end{aligned} \tag{3.10}$$

in order to make the value of the portfolio riskless, where r is a riskless rate, say bank account.

$$\begin{aligned} d\Phi(t) &= \left(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^{\xi} \frac{\partial^2 V}{\partial S^2} \right) dt \\ \therefore &+ \sigma S^{\frac{\xi}{2}} \frac{\partial V}{\partial S} dW - \Delta \left(\alpha S dt + \sigma S^{\frac{\xi}{2}} dW \right) \end{aligned} \tag{3.11}$$

$$\Rightarrow \begin{cases} \frac{d\Phi}{dt} = \left(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^{\xi} \frac{\partial^2 V}{\partial S^2} \right) \\ + \sigma S^{\frac{\xi}{2}} \frac{\partial V}{\partial S} \frac{dW}{dt} - \frac{\partial V}{\partial S} \left(\alpha S + \sigma S^{\frac{\xi}{2}} \frac{dW}{dt} \right) \end{cases}$$

showing that:

$$\frac{d\Phi}{dt} = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^{\xi} \frac{\partial^2 V}{\partial S^2} \equiv r\Phi. \tag{3.12}$$

Thus, using (3.10) in (3.12), we have:

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^{\xi} \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{3.13}$$

Hence, we refer to (3.13) as the GSGDM.

3.2 Comparison of the models

In this subsection, we shall compare the basic features of the two models presented above.

Suppose V_{BSM}^o denotes the volatility of the B-S model, V_{GSGDM}^o the volatility of the GSGDM associated SDE, V_{BSM}^r the variance of the B-S model, and V_{GSGDM}^r the variance of the GSGDM associated SDE. Then the following can easily be deduced:

$$\begin{aligned} V_{BSM}^o &= \sigma, \quad V_{GSGDM}^o = \sigma S^{\frac{\xi}{2}-1}, \quad V_{BSM}^r = \sigma^2, \text{ and} \\ V_{GSGDM}^r &= \sigma^2 S^{\xi-2}. \end{aligned} \tag{3.14}$$

Remark 3.2:

It is obvious that $V_{GSGDM}^o = h(\sigma, S_t)$ is not a constant function but a function of the underlying asset price S_t .

3.3: Special Cases of the GSGDM

In this subsection, we present the B-S and the square root models as special cases of the GSGDM.

3.3a: B-S Model and GSGDM

In (3.13), if $\xi = 2$ then the PDE representing the classical B-S model associated with the stochastic dynamics in (1.1) is obtained (see (1.2)).

3.3b: Square-root CEV Model and GSGDM

In (3.13), if $\xi = 1$ then the PDE model and the associated with the stochastic dynamics are:

$$\frac{\partial V}{\partial t} + rS \left(\frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) - rV = 0 \tag{3.15}$$

and

$$dS_t = S_t \mu dt + \sigma \sqrt{S_t} d(W_t) \tag{3.16}$$

respectively.

Equation (3.16) is referred to as the square-root CEV process dynamics [7, 15].

3.4 Modified VIM for the Generalized Square-root CEV Model

Comparing (3.15) with (3.1) makes it obvious that:

$$LV = V_t, Ru = \left\{ rS \left(V_S + \frac{1}{2} \sigma^2 V_{SS} \right) - rV \right\},$$

$$Nu = 0 \text{ and } f(x, t) = 0. \text{ Hence, for } I_{0,t} = \int_0^t (\cdot) ds,$$

we have:

$$V_{n+1} = V_n - I_{0,t} \begin{pmatrix} \left(rS \left(V_{S_n} + \frac{1}{2} \sigma^2 V_{SS_n} \right) - rV_n \right) \\ - \left(rS \left(V_{S_{n-1}} + \frac{1}{2} \sigma^2 V_{SS_{n-1}} \right) - rV_{n-1} \right) \end{pmatrix} \quad (3.17)$$

$$\begin{cases} V_{-1} = 0, \\ V_0 = V(S, 0), \\ V_1 = V_0. \end{cases} \quad (3.18)$$

Since, $Nu = 0$.

Thus, for $n \geq 1$, we have:

$$V_2 = V_1 - I_{0,t} \begin{pmatrix} \left(rS \left(V_{S_1} + \frac{1}{2} \sigma^2 V_{SS_1} \right) - rV_1 \right) \\ - \left(rS \left(V_{S_0} + \frac{1}{2} \sigma^2 V_{SS_0} \right) - rV_0 \right) \end{pmatrix},$$

$$V_3 = V_2 - I_{0,t} \begin{pmatrix} \left(rS \left(V_{S_2} + \frac{1}{2} \sigma^2 V_{SS_2} \right) - rV_2 \right) \\ - \left(rS \left(V_{S_1} + \frac{1}{2} \sigma^2 V_{SS_1} \right) - rV_1 \right) \end{pmatrix},$$

$$V_4 = V_3 - I_{0,t} \begin{pmatrix} \left(rS \left(V_{S_3} + \frac{1}{2} \sigma^2 V_{SS_3} \right) - rV_3 \right) \\ - \left(rS \left(V_{S_2} + \frac{1}{2} \sigma^2 V_{SS_2} \right) - rV_2 \right) \end{pmatrix},$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

So, $V_{n+1}(S, t) \rightarrow V(S, t)$ as $n \rightarrow \infty$, implies that:

$$V(S, t) = \lim_{k \rightarrow \infty} \left\{ V_{k-1} - I_{0,t} (H(S, t)) \right\}, k \geq 2. \quad (3.19)$$

where

$$H(S, t) = \begin{cases} \left(rS \left(V_{S_{k-1}} + \frac{1}{2} \sigma^2 V_{SS_{k-1}} \right) - rV_{k-1} \right) \\ - \left(rS \left(V_{S_{k-2}} + \frac{1}{2} \sigma^2 V_{SS_{k-2}} \right) - rV_{k-2} \right) \end{cases}$$

Equation (3.19) represents the general solution of the GSGDM.

5 CONCLUSION

In this paper, we derived via the application of Ito lemma and the CEV stochastic dynamics, a pricing model referred to as Generalized Squared Gaussian Diffusion Model (GSGDM) for option pricing. The key merit of this model is that the stock price volatility is a function of the underlying asset price; not a constant as noted in the assumptions of the classical Black-Scholes model. We further showed that the classical Black-Scholes, and the square root constant elasticity of variance models are special cases of the GSGDM. In addition, we obtained a general solution of the GSGDM using modified variational iterative method.

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