Derivation of minimum steel ratio: based on service applied stresses

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Abstract. Reinforced concrete beams undergo several design stages starting with the immediate application of loading and till the ultimate failure of the beam. In many practical applications, the beam is designed to function in the service stage during which steel plays a major role in the determination of the location of the Neutral axis. The ACI code implements several provisions on the minimum steel ratio in order to encourage ductile behaviour of the beam and to prevent against the unfavourable scenario of a sudden failure. The ACI 318-08 provisions encompass a derived expression to calculate the minimum amount of flexural reinforcement that is independent of concrete strength. This paper suggests an expansion to the formula by deriving the minimum steel ratio based on the modulus of rupture and the applied service stresses. The special case for a cracking bending moment is extrapolated from the suggested formula and the result is compared to different minimum steel area formulas.

1 Introduction

Tensile steel reinforcement is indispensable to the concrete beam should the applied loads produce stresses that exceed the modulus of rupture. Steel becomes active then providing the necessary tensile strength and forming the cracked transformed section of the beam. The transformed section consists of uncracked concrete that is limited to the area above the neutral axis and an "equivalent" area of concrete to the steel located below the neutral axis. Upon the determination of the depth of the neutral axis, the cracked moment of inertia of the beam is calculated, and consequently the stresses in both concrete and steel are calculated and compared to the applied service stress limits. The process is usually conducted with the area of steel given from the start or based on the ACI code provisions [1].

However, the equation provided according to the ACI code is intended of the case of a prevailing concrete strength. Salmon [2] have updated and expanded the formula to account for the modulus of rupture and introduced his equation into ACI 318-95 [3]. In addition to the ACI several codes, specifications and researches have come up with their criteria for the minimum steel area. Some of these are listed in Table 1.

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Table 1. Minimum Steel area for reinforced concrete.

Source	Year	Minimum Steel Area	Comments
Proposed	2017	$A_{s,min} = \frac{{k_{\min}}^2}{2n(1-k_{\min})} b_w d_s$ $k_{\min} = \max\begin{cases} k_1 \\ k_4 \end{cases}$	$k_1 = \frac{3-\sqrt{9-4n_c n_h^2}}{2}$ $k_4 = 1 + C_n(\sqrt{3}sin\alpha - cos\alpha)$
ACI 318-08[3]	1963	$A_{s,min} = \frac{3\sqrt{\acute{f}_c}}{f_y} b_w d_s$ $\geq \frac{200}{f_y} b_w d_s$	
Freyermuth and Aalami [4]	1997	$A_{s,min} = \frac{3\sqrt{\acute{f}_c}}{f_{su}} b_w d_s$	
ASBI [5]	1964	$A_{s,min} = \frac{0.3 f_{ct}}{f_y} b_w h$	$f_{ct} = 7.3\sqrt{\acute{f}_c}$
Salmon[2]	1995	$A_{s,min} = \frac{f_r}{f_y}\left(\frac{h}{d_s}\right)^2\left(\frac{C}{5.1}\right) b_w d_s$	C is a multiplier hat adjusts the section modulus for different beam shapes

Note: $A_{s,min}$ = minimum area of nonprestressed flexural tension reinforcement; h = height of the section; b_w = web width; d_s = distance from extreme compression fiber to centroid of nonprestressed flexural tension reinforcement; $\acute{f}_c$ = specified compressive strength of concrete; f_{ct} = direct tensile strength of concrete; f_{su} = specified tensile strength of nonprestressed flexural tension reinforcement; f_r = modulus of rupture of concrete; f_y = specified minimum yield strength of nonprestressed flexural tension reinforcement; n_c = is the ratio of modulus of rupture to allowable compressive strength; n_s = is the ratio of modulus of rupture to allowable tensile strength; n_h = is the ratio of section height to depth of reinforcement; C_n and α are variables defined in the context of this paper.

2 Methodology of derivation of minimum steel ratio

The derivation of minimum steel ratio in this paper proceeds in the following order:

a. Derivation of the Neutral axis depth ratio ($\boldsymbol{k}$) and solving for the corresponding steel ratio
b. Expressing the cracked moment of inertia in terms of $\boldsymbol{k}$
c. Solving for $\boldsymbol{k}$ in the flexural stress formula of concrete and steel and dismissing the unrealistic/unfeasible solutions
d. Extrapolating the special cracking moment case and obtaining the minimum feasible limit of $\boldsymbol{k}$
e. Substituting $\boldsymbol{k}$ in the steel ratio formula and solving to obtain expression for minimum steel ratio

The derived formula assumes a linear stress-strain relation and a perfect bond between the concrete and steel. However, the procedure is limited to the case of a rectangular beams or T beams with neutral axis in the flange.

3 Location of neutral axis

Once the beam has beam cracked, the neutral axis of the beam immediately deviates from the centroid of the beam, and the transformed section is formulated as shown in Figure 1**.** The neutral axis location is defined as the point of which the first moment of the section about the neutral axis is zero. This leads to the formulation of the following Equation 1:

$$b\frac{x^2}{2} = n\rho bd(d - x) \qquad (1)$$

Solving for x leads to: -

$$x = \left(\sqrt{(n\rho)^2 + 2n\rho} - n\rho\right)d = kd \qquad (2)$$

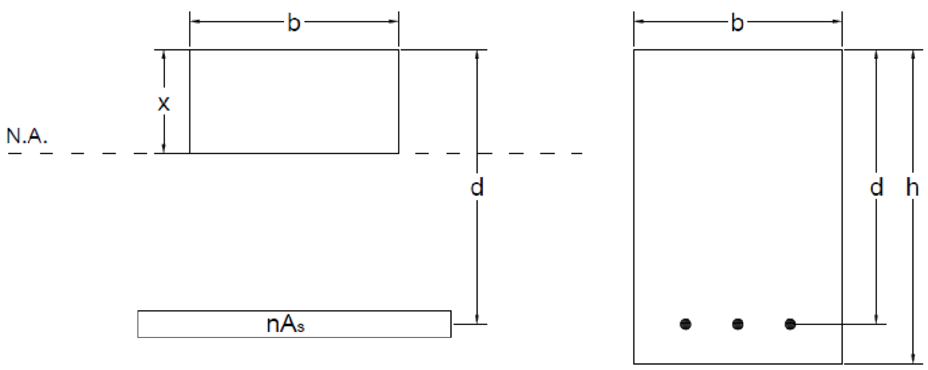

Fig. 1. Transformed area of the cross-section.

Equation 2 determines the depth of the neutral axis as a some ratio k (called the N.A ratio) multiplied by the depth of the reinforcement steel d. Solving Equation 2 for steel ratio result in Equation 3: -

$$\rho = \frac{k^2}{2n(1-k)} \qquad (3)$$

Despite the fact that k is limited to a maximum value of 1, the steel ratio in unbounded, this is evident in Figure 2 as shown.

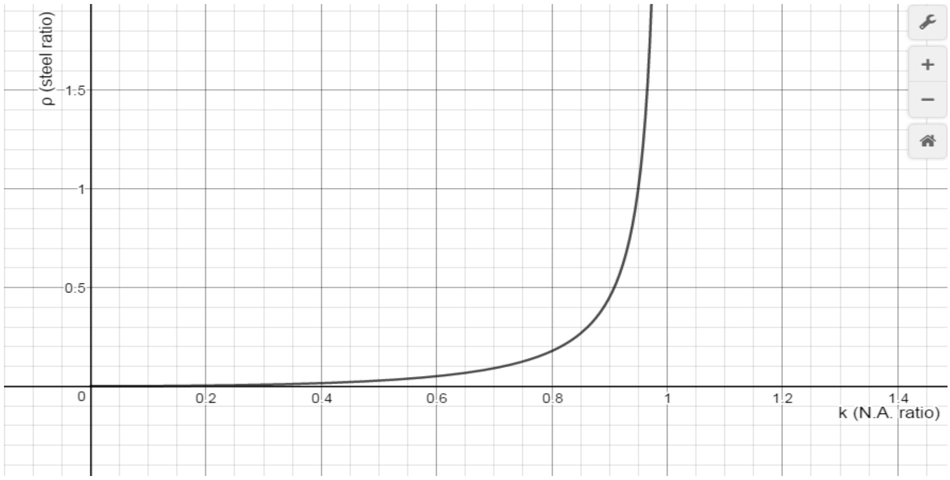

Fig. 2. Relationship between steel ratio and N.A. ratio.

4 Evaluating cracked moment of inertia

The cracked moment of inertia is then evaluated according to Equation 5 which is obtained by substituting Equation 2 and Equation 3 into Equation 4: -

$$I_{cr} = \frac{1}{3}bx^3 + n\rho bd(d-x)^2 \tag{4}$$

$$I_{cr} = \left(\frac{k^2}{2} - \frac{k^3}{6}\right) bd^3 \tag{5}$$

5 Applying the concrete service stress limit

The stresses applied in the extreme compression fibers of concrete must be limited to the service stress limit as displayed according to Equation 6: -

$$f_c \geq \frac{M_c x}{I_{cr}} \tag{6}$$

Substituting Equation 2 and Equation 5 into Equation 6 and rearranging the terms, a quadratic form in terms of k results as Equation 6 dictates: -

$$k^2 - 3k + \frac{6M_c}{f_c bd^2} \leq 0 \tag{7}$$

Solution of Equation 7 produces two solutions:

$$k_1 = \frac{3 - \sqrt{9 - \frac{24M_c}{f_c bd^2}}}{2} \tag{8}$$

$$k_2 = \frac{3 + \sqrt{9 - \frac{24M_c}{f_c bd^2}}}{2} \tag{9}$$

The second solution is dropped because it yields k that is always greater than one. Therefore, the feasible solutions of the inequality presented in Equation 7 lie within the range $[k_1, 1]$. It must be noticed that while the interval $[k_1, 1]$ is a solution to the inequality presented in Equation 7, the interest of this paper is the value of k_1 itself because according to Figure 2 the minimum feasible value of k, results in the corresponding minimum feasible value of ρ. Furthermore, by imposing the limitation that $f_c \geq f_r$, it can be shown that k_1 is limited to not exceed an approximate value of 0.53.

This obtained feasible range however, is based solely on the compression limit of the concrete. Thus, the steel tensile limit must be treated similarly.

6 Applying the steel service stress limit

The stresses applied in the tensile reinforcement steel must be limited to the service stress limit as displayed according to Equation 10: -

$$f_s \geq \frac{M_s(d-x)n}{I_{cr}} \quad (10)$$

Substituting Equation 2 and Equation 5 into Equation 10 and rearranging the terms, a cubic form in terms of k results as Equation 11 dictates: -

$$k^3 - 3k^2 - \frac{6M_s n}{f_s b d^2}k + \frac{6M_s n}{f_s b d^2} \leq 0 \quad (11)$$

Solution of **Equation 11** produces three solutions: -

$$k_3 = 1 + 2C_n cos\alpha \quad (12)$$

$$k_4 = 1 + C_n\left(\sqrt{3}sin\alpha - cos\alpha\right) \quad (13)$$

$$k_5 = 1 - C_n\left(\sqrt{3}sin\alpha + cos\alpha\right) \quad (14)$$

Where

$$C_n = \sqrt{1 + \frac{2M_s n}{f_s b d^2}} \quad (15)$$

$$\alpha = \frac{1}{3}arcsin\sqrt{\frac{\left(1+\frac{2M_s n}{f_s b d^2}\right)^3 - 1}{\left(1+\frac{2M_s n}{f_s b d^2}\right)^3}} \quad (16)$$

The first solution is dropped because it yields k that is always greater than one. The last solution is also dropped because it turns out to be negative for all practical cases. Therefore, the feasible solutions of the inequality presented in Equation 11 lie within the range $[k_4, 1]$.

Likewise, the interest of this paper is the value of k_4 itself because it is less than 1. The value of k_4 is then compared to the value of k_1 and the maximum of the two is considered to control as it satisfies both of inequalities Equation 7 and Equation 11.

7 Extrapolating the special case of the cracking moment

Modifying both Equations 8 and 13 for the cracking moment defined according to ACI in Equation 17 results in the following: -

$$M_{cr} = \frac{f_r \cdot bh^3}{6} \tag{17}$$

$$k_1 = \frac{3 - \sqrt{9 - 4n_c n_h^2}}{2} \tag{18}$$

$$k_4 = 1 + C_n\left(\sqrt{3}sin\alpha - cos\alpha\right) \tag{19}$$

Where

$$C_n = \sqrt{1 + \frac{nn_s n_h^2}{3}} \tag{20}$$

$$\alpha = \frac{1}{3}arcsin\sqrt{\frac{\left(1+\frac{nn_s n_h^2}{3}\right)^3 - 1}{\left(1+\frac{nn_s n_h^2}{3}\right)^3}} \tag{21}$$

$$n_c = \frac{f_r}{f_c} \tag{22}$$

$$n_s = \frac{f_r}{f_s} \tag{23}$$

$$n_h = \frac{h}{d} \tag{24}$$

This formulation presented in Equations 18-24 renders the definition of the minimum k dependent upon the three dimensionless ratios $\mathrm{n_c}$, $\mathrm{n_s}$, and $\mathrm{n_h}$ exclusively. k_{min} is then defined as: -

$$k_{min} = max\begin{cases} k_1 \\ k_4 \end{cases} \tag{25}$$

Finally, the minimum steel ratio is computed as: -

$$\rho_{min} = \frac{{k_{min}}^2}{2n(1-k_{min})} \tag{26}$$

Accordingly, the minimum area of steel is determined by multiplying ρ_{min} with the section width and effective depth.

8 Comparison of proposal with other formulas

In order to validate the derivation of the minimum steal ratio formula that has been provided in this paper, a comparison to other models is made. The comparison assumes fixed parameters as indicated in Table 2 and evaluates the minimum area of steal according to each formula that has been presented in Table 1. Accordingly, the results are tabulated as shown in Tables 3 and 4.

Table 2. Section geometric and material properties of the beam.

Geometry			Material		
Section property	**Dimension**	**Unit**	**Material property**	**Dimension**	**Unit**
$\boldsymbol{h}$	20	in	$\boldsymbol{f_y = f_s}$	60000 / 40000	psi
$\boldsymbol{b}$	12	in	$\boldsymbol{f_r}$	$7.5\sqrt{\acute{f}_c}$	psi
$\boldsymbol{d}$	17	in	$\boldsymbol{f_{ct}}$	$7.3\sqrt{\acute{f}_c}$	psi
$\boldsymbol{n}$	9	-	$\boldsymbol{f_c}$	$\frac{1}{2}\acute{f}_c$	psi

Table 3. Minimum steel area results in in^2 for $f_y = 60000\ psi$.

$\boldsymbol{\acute{f}_c\ (psi)}$	**1500**	**2000**	**2500**	**3000**	**3500**	**4000**	**4500**	**5000**	**5500**	**6000**
ACI 318-08	0.680	0.680	0.680	0.680	0.680	0.680	0.684	0.721	0.756	0.790
	0	0	0	0	0	0	2	2	5	1
ASBI	0.339	0.391	0.438	0.479	0.518	0.554	0.587	0.619	0.649	0.678
	3	8	0	8	2	0	6	4	7	5
Salmon	0.268	0.309	0.346	0.379	0.409	0.437	0.464	0.489	0.513	0.536
	0	5	0	0	4	7	2	3	2	1
Proposed	0.510	0.363	0.309	0.340	0.368	0.394	0.419	0.442	0.464	0.485
	0	0	8	2	2	4	0	3	6	9
Freyermuth and Aalami	0.379	0.437	0.489	0.536	0.579	0.619	0.656	0.692	0.726	0.758
	2	9	6	3	3	3	9	4	2	5

Table 4. Minimum steel area results in in^2 for $f_y = 40000\ psi$.

$\acute{f}_c\ (psi)$	1500	2000	2500	3000	3500	4000	4500	5000	5500	6000
ACI 318-08	1.0200	1.0200	1.0200	1.0200	1.0200	1.0200	1.0264	1.0819	1.1347	1.1851
ASBI	0.5089	0.5876	0.6570	0.7197	0.7774	0.8310	0.8815	0.9291	0.9745	1.0178
Salmon	0.4020	0.4642	0.5190	0.5686	0.6141	0.6565	0.6964	0.7340	0.7698	0.8041
Proposed	0.5100	0.4190	0.4700	0.5163	0.5591	0.5990	0.6366	0.6723	0.7063	0.7388
Freyermuth and Aalami	0.5689	0.6569	0.7344	0.8045	0.8690	0.9290	0.9853	1.0386	1.0893	1.1377

The results are plotted against each other in **Figures 3 and 4**.

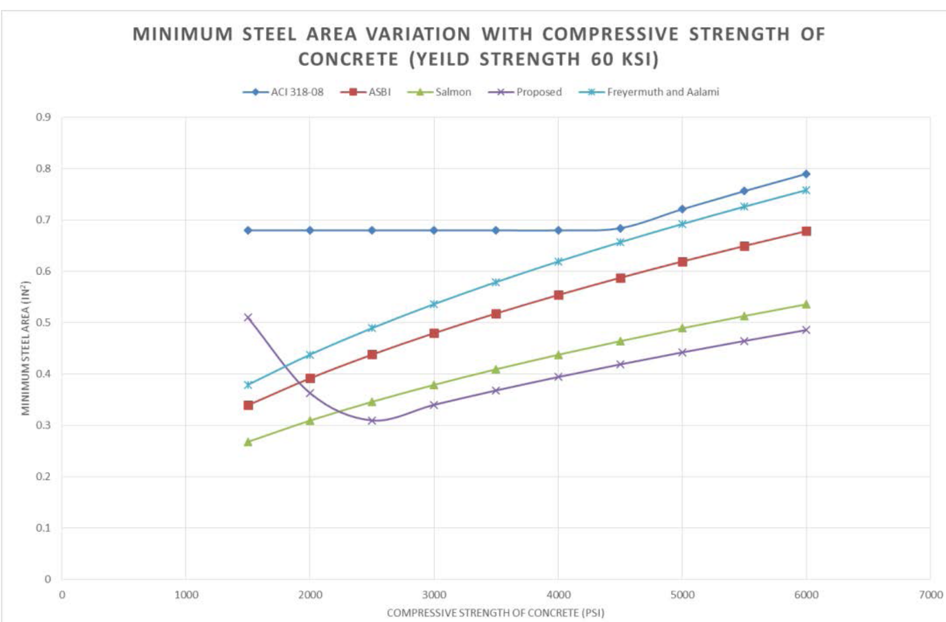

Fig. 3. Results for a yield strength of steel $f_y = 60000\ psi$.

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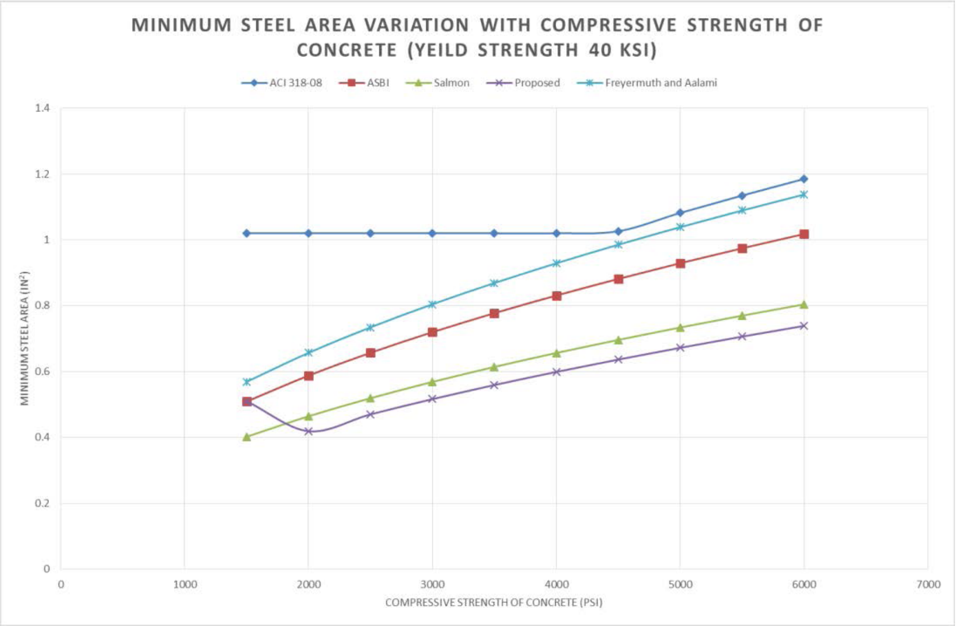

Fig. 4: Results for a yield strength of steel $f_y = 40000\, psi$

9 Discussion

The pattern of the results as displayed in Figure 3 indicates complete equivalence in the general trend of the minimum steal area resulting from all the formula, with the ACI 318-08 being the most conservative of all. Among the investigated formulas, only two of them are defined as a compound function of two criteria: steel and concrete stress requirement. Namely, the ACI-318-08 and the proposed formula. From the comparison, it can be noted that the first criteria of the ACI-318-08 formula namely $\frac{200}{f_y}\, b_w d_s$ is far more conservative than the second criteria.

Furthermore, the proposed formula seems to deviate from the general trend at values of concrete compressive strength roughly around $2000\, psi$, throughout which the concrete stress limit controls. It should be noted that the proposed formula is the only one that clearly accounts for both the stress limits of concrete and steel resulting in the possibility to define the best match of concrete and steel stress limits. **Figures 3 and 4** indicate that $\mathrm{f_y} = 40000\,\mathrm{psi}$ is matching more harmonically with the results as the trend of the proposed formula exhibits less disparity when compared to $\mathrm{f_y} = 60000\,\mathrm{psi}$.

10 Conclusion

The proposed formula is clearly in match with the other formulas but it results in a minimum area of steel that is less than the ones resulting from all the other formulas. The computations of the proposed formula is a little more complicated, however, this issue is of no concern to programs as the formula can be computed easily in a straight forward algorithm without any need for any sort of cyclic routines. The formula is however (like the rest), limited to the assumptions presented in the methodology section. Further work is expected to expand the range of applicability of the proposed formula.

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