Calculation of the filtration of polydisperse suspension with a small rate

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Abstract. The study of filtration problem is important when designing and constructing structures on water saturated ground. The filtration of a suspension with particles of different sizes moving at the same speed in a homogeneous porous medium is considered. A global asymptotic solution is constructed for a small filtration coefficient. The numerical calculation is based on experimental data.

1. Introduction

The study of filtration of suspended particulate matter - suspension in a porous medium is an important problem of underground hydromechanics [1]. For a long-term evaluation of the foundations strength it is necessary to take into account the composition of the soil and the penetrating capacity of groundwater.

A model of deep bed filtration of a suspension with suspended particles of various sizes moving with the same velocities in a homogeneous porous medium is considered. It is assumed that the particle and pore size distributions overlap, and the dominant cause of particle capture is the size-exclusion [2]. The suspended particles pass freely through large pores and are stuck at the inlet of pores, smaller than the diameter of the particles. One particle completely blocks the small pore and can not be knocked out of its throat by a flow of carrier fluid or other particles.

The mathematical model of filtration includes the equation of mass balance of suspended and retained particles, which is an analog of the continuity equation, and also the kinetic equation describing the rate of deposit formation proportional to the concentration of suspended particles [3]. The coefficient of proportionality, called the filtration coefficient, depends on the value of the deposit. As the deposit forms, the amount of free small pores decreases, and the growth rate of the deposit slows down.

Two quasilinear differential equations form a hyperbolic system. The boundary conditions that uniquely determine the solution of the system are set at the filter inlet and at the initial time. It is assumed that at the initial moment the porous medium does not contain any particles. A suspension containing n different types of suspended particles of constant concentration is injected at the filter inlet.

A number of filtration problems have an exact analytical solution [4-8]. In the absence of an exact solution, asymptotics are constructed [9-12], or the problem is solved

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numerically [13-15]. In the paper an asymptotic solution of the filtration problem of a polydisperse suspension in a porous medium is constructed assuming that the filtration coefficient is small. Numerical calculation of the asymptotics at the filter outlet is performed for 3-size particles on the basis of experimental data [10].

2. The mathematical model

In the domain

$$\Omega = \{0 < x < 1, t > 0\}$$

consider a system of equations with respect to the concentrations of suspended C_i and retained S_i particles

$$\frac{\partial C_i}{\partial t} + \frac{\partial C_i}{\partial x} + \frac{\partial S_i}{\partial t} = 0; \quad (1)$$

$$\frac{\partial S_i}{\partial t} = \varepsilon \Lambda_i(\vec{S}) C_i; \quad i = 1, \dots, n \quad (2)$$

with the boundary condition

$$x = 0: C_i = p_i \quad (3)$$

and the initial conditions

$$t = 0: C_i = 0, S_i = 0. \quad (4)$$

Here $\varepsilon > 0$ is a small parameter, $\vec{S} = (S_1, \dots, S_n)$; the filtration coefficients $\Lambda_i(\vec{S})$ are continuous and positive; the constants $p_i > 0$.

The concentrations front of the suspended and retained particles moves from the filter inlet $x = 0$ to the outlet $x = 1$ at a constant velocity $v = 1$ and divides the domain Ω into two subdomains $\Omega_0 = \{0 < x < 1, 0 < t < x\}$ and $\Omega_s = \{0 < x < 1, t > x\}$. In Ω_0 the system (1) – (4) has a zero solution; in Ω_s the solution is positive. Since conditions (3) and (4) are not matched at the origin, the solution has a strong discontinuity on the concentration front -- the characteristic line $t = x$. The solution $S_i(x, t)$ is continuous in Ω and has a weak discontinuity on the concentration front.

With increasing time t , the concentration of retained particles increases, and the number of free small pores is reduced. The growth rate of the deposit slows down and the suspended particles concentration increases:

$$\frac{\partial S_i}{\partial t} > 0; \quad \frac{\partial C_i}{\partial t} > 0. \quad (5)$$

The retained particles concentration decreases with increasing distance x from the porous medium entrance, since the filtration process of the suspension starts at the inlet of the filter. With a smaller deposit, the particle retention occurs more intensively, and the concentration of suspended particles also decreases with increasing x .

$$\frac{\partial S_i}{\partial x} < 0; \quad \frac{\partial C_i}{\partial x} < 0. \quad (6)$$

Consider the condition on the characteristic line

$$S_i|_{t=x} = 0. \quad (7)$$

In the domain Ω_s the problem (1) – (4) is equivalent to the Goursat problem (1) – (3), (7).

In the characteristic variables (Riemann variables)

$$\tau = t - x, \quad x = x$$

in the domain $\tilde{\Omega}_s = \{0 < x < 1, \tau > 0\}$ the system (1) – (3), (7) takes the form

$$\frac{\partial C_i}{\partial x} + \frac{\partial S_i}{\partial \tau} = 0; \quad (8)$$

$$\frac{\partial S_i}{\partial \tau} = \varepsilon \Lambda_i(\vec{S}) C_i \quad (9)$$

with the conditions

$$x = 0: C_i = p_i; \quad (10)$$

$$\tau = 0: S_i = 0; i = 1, \dots, n. \quad (11)$$

3. The asymptotic solution of the filtration problem

Assume that the filtration coefficients $\Lambda_i(\vec{S})$ have the form

$$\Lambda_i(\vec{S}) = \lambda_i + \sum_{m=1}^n a_i^m S_m + \frac{1}{2} \sum_{k,l=1}^n b_i^{k,l} S_k S_l + O(|\vec{S}|^3); \quad (12)$$

$$\lambda_i = \Lambda_i(\vec{0}), \quad a_i^m = \frac{\partial \Lambda_i}{\partial S_m}(\vec{0}) > 0, \quad b_i^{k,l} = \frac{\partial^2 \Lambda_i}{\partial S_k \partial S_l}(\vec{0}), \quad \vec{0} = (0, 0, \dots, 0), \quad i = 1, \dots, n.$$

In the domain $\tilde{\Omega}_s$, the asymptotics of the problem (8) – (11) is constructed in the form

$$S_i = \varepsilon s_i^1 + \varepsilon^2 s_i^2 + \varepsilon^3 s_i^3 + O(\varepsilon^4); \quad C_i = p_i + \varepsilon c_i^1 + \varepsilon^2 c_i^2 + \varepsilon^3 c_i^3 + O(\varepsilon^4); \quad i = 1, \dots, n. \quad (13)$$

Substituting the expansions (12), (13) into equations (8), (9) and equating terms for identical powers of the small parameter ε , we obtain a recurrent system of differential equations

$$\frac{\partial c_i^1}{\partial x} + \lambda_i p_i = 0; \quad (14)$$

$$\frac{\partial s_i^1}{\partial \tau} = \lambda_i p_i; \quad (15)$$

$$\frac{\partial c_i^2}{\partial x} + p_i \sum_{m=1}^n a_i^m s_m^1 + \lambda_i c_i^1 = 0; \quad (16)$$

$$\frac{\partial s_i^2}{\partial \tau} = p_i \sum_{m=1}^n a_i^m s_m^1 + \lambda_i c_i^1; \quad (17)$$

$$\frac{\partial c_i^3}{\partial x} + p_i \sum_{m=1}^n a_i^m s_m^2 + \frac{p_i}{2} \sum_{k,l=1}^n b_i^{k,l} s_k^1 s_l^1 + \sum_{m=1}^n a_i^m s_m^1 c_i^1 + \lambda_i c_i^2 = 0; \quad (18)$$

$$\frac{\partial s_i^3}{\partial \tau} = p_i \sum_{m=1}^n a_i^m s_m^2 + \frac{p_i}{2} \sum_{k,l=1}^n b_i^{k,l} s_k^1 s_l^1 + \sum_{m=1}^n a_i^m s_m^1 c_i^1 + \lambda_i c_i^2. \quad (19)$$

The initial conditions for the asymptotic terms follow from (3), (7):

$$c_i^j \Big|_{x=0} = 0; \quad s_i^j \Big|_{\tau=0} = 0; \quad i = 1, \dots, n; \quad j = 1, 2, 3. \quad (20)$$

The terms of the asymptotic expansions are obtained by successive solution of the equation (14) – (19) with conditions (20)

$$c_i^1 = -\lambda_i p_i x; \quad (21)$$

$$s_i^1 = \lambda_i p_i \tau; \quad (22)$$

$$c_i^2 = \lambda_i^2 p_i \frac{x^2}{2} - p_i A_i^1 \tau x; \quad (23)$$

$$s_i^2 = -\lambda_i p_i A_i^1 \frac{\tau^2}{2} - \lambda_i^2 p_i \tau x. \quad (24)$$

$$c_i^3 = -p_i (A_i^3 + B_i) x \frac{\tau^2}{2} + p_i (A_i^2 + 2\lambda_i A_i^1) \frac{x^2}{2} \tau - \lambda_i^3 p_i \frac{x^3}{6}; \quad (25)$$

$$s_i^3 = p_i (A_i^3 + B_i) \frac{\tau^3}{6} - p_i (A_i^2 + 2\lambda_i A_i^1) x \frac{\tau^2}{2} + \lambda_i^3 p_i \frac{x^2}{2} \tau; \quad (26)$$

where

$$A_i^1 = \sum_{m=1}^n a_i^m \lambda_m p_m; \quad A_i^2 = \sum_{m=1}^n a_i^m \lambda_m^2 p_m; \quad A_i^3 = \sum_{m=1}^n a_i^m A_m^1 p_m; \quad B_i = \sum_{k,l=1}^n b_i^{k,l} \lambda_k \lambda_l p_k p_l; \quad i = 1, \dots, n.$$

Substituting the solutions (19) - (26) into the expansions (13) and performing an inverse change of variables, we find the asymptotic solution of the problem (8) – (11) in Ω_S :

$$\begin{aligned} C_i(x, t) = & p_i - \lambda_i p_i x \varepsilon + p_i x \left(\lambda_i^2 \frac{x}{2} - A_i^1 (t - x) \right) \varepsilon^2 - \\ & - p_i x \left((A_i^3 + B_i) \frac{(t - x)^2}{2} - (A_i^2 + 2\lambda_i A_i^1) \frac{x}{2} (t - x) + \lambda_i^3 \frac{x^2}{6} \right) \varepsilon^3 + O(\varepsilon^4); \end{aligned} \quad (27)$$

$$S_i(x, t) = \lambda_i p_i (t - x) \varepsilon - p_i \left(-A_i^1 \frac{(t - x)^2}{2} + \lambda_i^2 (t - x) x \right) \varepsilon^2 + \\ + p_i (t - x) \left((A_i^3 + B_i) \frac{(t - x)^2}{6} - (A_i^2 + 2\lambda_i A_i^1) x \frac{(t - x)}{2} + \lambda_i^3 \frac{x^2}{2} \right) \varepsilon^3 + O(\varepsilon^4); \quad i = 1, \dots, n. \quad (28)$$

The applicability of the asymptotics (27), (28) is determined by the inequalities (5), (6).

4. Numerical simulation

The calculation is performed for spherical particles of 3 radii $r_1 = 1.5675 \mu\text{m}$, $r_2 = 2.179 \mu\text{m}$ and $r_3 = 3.168 \mu\text{m}$, for the small parameter $\varepsilon = 0.25$. The concentrations of suspended particles at the filter inlet $p_1 = 1$, $p_2 = 0.2$, $p_3 = 0.07$ are chosen so that the masses of the 3 types of injected suspended particles are the same. The filtration coefficients were chosen in accordance with experimental data [10]

$$\Lambda_1(S) = 0.110 - 0.014S + 4,49 \cdot 10^{-5} S^2,$$

$$\Lambda_2(S) = 0.510 - 0.00596S + 2,29 \cdot 10^{-6} S^2,$$

$$\Lambda_3(S) = 1.551 - 0.00347S - 1,16 \cdot 10^{-6} S^2.$$

Here $S = S_1 + S_2 + S_3$ is the total retained particles concentration.

Fig. 1 shows the graphs of suspended particles concentrations for 3-size particles a) at the filter outlet and b) at $t = 0.5$ (the smaller the particle size, the higher the graph). The dashed line indicates the maximum value $C = 1$.

Fig. 1. a) The graphs of $C_i(x, t)|_{x=1}$.

b) The graphs of $C_i(x, t)|_{t=0.5}$.

Fig. 2 presents the graphs of the retained particles concentrations for the particles of 3 sizes a) at time $t = 0.5$ and b) at time $t = 5$ (at the filter inlet $x = 0$ the graphs are higher for larger particles).

Fig. 2. a) The graphs of $S_i(x, t)|_{t=0.5}$.b) The graphs of $S_i(x, t)|_{t=5}$.

5. Conclusion

The global asymptotic solution of the one-dimensional filtration problem of a polydisperse suspension in a homogeneous porous medium for small filtration coefficients is constructed. In the paper the methods of [16] are extended to the case of n -size particles. The solution has a break on the characteristic line. Fig. 1 a) and b) show a gap in the suspended particles concentration on the concentration front $t = x$. On the graph of the retained particles concentration there is a kink at the point of intersection with the concentration front $x = 0.5$ (Fig. 2 a)).

The rate of decrease in the retained particles concentration with increasing distance to the filter inlet depends on the particle size (Fig. 2 b)). The larger the particle, the faster the rate of decrease. The graphs confirm the experimental fact that larger particles mostly get stuck near the filter inlet, and fine particles are retained uniformly throughout the entire porous medium.

The obtained asymptotic solution improves the qualitative tuning of the mathematical model and reduces the number of laboratory experiments [17].

Mathematical methods used to construct the asymptotics can be generalized to the case of variable porosity and permeability, depending on the concentration of retained particles [18].

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