

# About solution of multipoint boundary problem of static analysis of deep beam with the use of combined application of finite element method and discrete-continual finite element method. part 2: boundary conditions

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**Abstract.** The formulation of considering multipoint boundary problem includes three main components: a description of the domain occupied by the structure and the corresponding subdomains; description of the conditions inside the domain and inside subdomains; description of boundary conditions (for boundaries of domain and boundaries between subdomains). These boundary conditions (interface conditions) in under consideration in the distinctive paper.

## 1 Analysis of options for boundary conditions

In practical applications the following variants of boundary conditions (interface conditions) between subdomains (2) from the first part of this paper are most often encountered (some of them are considered in this paper): interface “discrete-continual model – discrete model”, “internal” boundary condition of the type “perfect contact”; interface “discrete model – discrete-continual model”, “internal” boundary condition of the type “perfect contact”; interface “discrete-continual model – discrete-continual model”, “internal” boundary condition of the type “perfect contact”; interface “discrete model – discrete model”, “internal” boundary condition of the type “perfect contact”; interface “external boundary – discrete model”, boundary condition of the type “hinged support”; interface “external boundary – discrete model”, boundary condition of the type “free edge”; interface “external boundary – discrete-continual model”, boundary condition of the type “hinged support”; interface “external boundary – discrete-continual model”, boundary condition of the type “free edge”; interface “discrete model – external boundary”, boundary condition of the type “hinged support”; interface “discrete model – external boundary”, boundary condition of the type “free edge”; interface “discrete-continual model – external boundary”, boundary condition of the type “hinged support”; interface “discrete-continual model –

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external boundary, boundary condition of the type “free edge”. Of course, other variants of interface are possible, but similarly, somehow, in one way or another, as a rule, these variants are reduced to some combinations of the above-mentioned twelve [6-16].

## 2 Interface “discrete-continual model – discrete model”, “internal” boundary condition of the type “perfect contact”

Let’s consider arbitrary boundary point  $x_{2,k}^b$ ,  $1 < k < n_b$  and corresponding interface between subdomain  $\omega_{k-1}^{dc}$  (subdomain is approximated with the use of DCFEM,  $\rho_{k-1} = 2$ ) and subdomain  $\omega_k^{fe}$  (subdomain is approximated with the use of FEM,  $\rho_k = 1$ ).

Boundary conditions at section  $x_2 = x_{2,k}^b$  (perfect contact) have form ( $4N_1$  equations):

$$u_1^{(k-1,i)}(x_{2,k}^b - 0) = u_1^{(k,i,j)}, \quad i = 1, 2, \dots, N_1, \quad j = 1; \tag{1}$$

$$u_2^{(k-1,i)}(x_{2,k}^b - 0) = u_2^{(k,i,j)}, \quad i = 1, 2, \dots, N_1, \quad j = 1; \tag{2}$$

$$\sigma_{1,2}^{(k-1,i)}(x_{2,k}^b - 0) = \sigma_{1,2}^{(k,i,j)}, \quad i = 1, 2, \dots, N_1, \quad j = 1; \tag{3}$$

$$\sigma_{2,2}^{(k-1,i)}(x_{2,k}^b - 0) = \sigma_{2,2}^{(k,i,j)}, \quad i = 1, 2, \dots, N_1, \quad j = 1, \tag{4}$$

where  $\sigma_{1,2}^{(k-1,i)}(x_{2,k}^b - 0)$  and  $\sigma_{2,2}^{(k-1,i)}(x_{2,k}^b - 0)$  are nodal (node  $(k, i)$ ) values of functions of stress components  $\sigma_{1,2}(x_2)$  and  $\sigma_{2,2}(x_2)$  (after corresponding averaging) for discrete-continual finite element  $\omega_{k-1}^{dc}$ ;  $\sigma_{1,2}^{(k,i,j)}$  and  $\sigma_{2,2}^{(k,i,j)}$  are nodal (node  $(k, i, j)$ ) stress components  $\sigma_{1,2}$  and  $\sigma_{2,2}$  (after corresponding averaging).

Equations (1)-(4) can be rewritten in matrix form:

$$B_k^- \bar{U}_{k-1}(x_{2,k}^b - 0) = B_k^+ \bar{U}_k, \tag{5}$$

where  $B_k^-$  is matrix of boundary conditions of size  $4N_1 \times 4N_1$ , which can be constructed in accordance with algorithm presented at Table 1;  $B_k^+$  is matrix of boundary conditions of size  $4N_1 \times 2N_1N_{2,k}$ , which can be constructed in accordance with so-called method of basis variations [1-5,17].

**Table 1.** Algorithm of computing of nonzero elements of matrix  $B_k^-$ .

Numbers (indexes) of elements	Element value	Corresponding boundary condition
1	2	3
$(i, 2i - 1), \quad i = 1, 2, \dots, N_1$	1	(1)
$(N_1 + i, 2i), \quad i = 1, 2, \dots, N_1$	1	(2)
$(2N_1 + 1, 2)$	$\bar{\mu}_{k-1,1} \frac{1}{h_1} N_1'(0)$	(3) $i = 1$
$(2N_1 + 1, 4)$	$\bar{\mu}_{k-1,1} \frac{1}{h_1} N_2'(0)$	

1	2	3
$(2N_1 + 1, 2N_1 + 1)$	$\bar{\mu}_{k-1,1}$	(3) $i = 1$
$(2N_1 + i, 2(i-1)),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\mu}_{k-1,i-1} \frac{1}{h_{i-1}} N'_1(1)$	(3) $i = 2, 3, \dots,$ $N_1 - 1$
$(2N_1 + i, 2i),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \left[ \frac{\bar{\mu}_{k-1,i-1}}{h_{i-1}} N'_2(1) + \frac{\bar{\mu}_{k-1,i}}{h_i} N'_1(0) \right]$	
$(2N_1 + i, 2(i+1)),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\mu}_{k-1,i} \frac{1}{h_i} N'_2(0)$	
$(2N_1 + i, 2(N_1 + i) - 1),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} (\bar{\mu}_{k-1,i-1} + \bar{\mu}_{k-1,i})$	
$(3N_1, 2N_1 - 2)$	$\bar{\mu}_{k-1,N_1-1} \frac{1}{h_{N_1-1}} N'_1(1)$	
$(3N_1, 2N_1)$	$\bar{\mu}_{k-1,N_1-1} \frac{1}{h_{N_1-1}} N'_2(1)$	
$(3N_1, 4N_1 - 1)$	$\bar{\mu}_{k-1,N_1-1}$	
$(3N_1 + 1, 1)$	$\bar{\lambda}_{k-1,1} \frac{1}{h_1} N'_1(0)$	(4) $i = 1$
$(3N_1 + 1, 3)$	$\bar{\lambda}_{k-1,1} \frac{1}{h_1} N'_2(0)$	
$(3N_1 + 1, 2N_1 + 1)$	$\bar{\lambda}_{k-1,1} + 2\bar{\mu}_{k-1,1}$	
$(3N_1 + i, 2i - 3),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\lambda}_{k-1,i-1} \frac{1}{h_{i-1}} N'_1(1)$	(4) $i = 2, 3, \dots,$ $N_1 - 1$
$(3N_1 + i, 2i - 1),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \left[ \frac{\bar{\lambda}_{k-1,i-1}}{h_{i-1}} N'_2(1) + \frac{\bar{\lambda}_{k-1,i}}{h_i} N'_1(0) \right]$	
$(3N_1 + i, 2i + 1),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\lambda}_{k-1,i} \frac{1}{h_i} N'_2(0)$	
$(3N_1 + i, 2(N_1 + i)),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} (\bar{\lambda}_{k-1,i-1} + \bar{\lambda}_{k-1,i}) +$ $\bar{\mu}_{k-1,i-1} - \bar{\mu}_{k-1,i}$	
$(4N_1, 2N_1 - 3)$	$\bar{\lambda}_{k-1,N_1-1} \frac{1}{h_{N_1-1}} N'_1(1)$	
$(4N_1, 2N_1 - 1)$	$\bar{\lambda}_{k-1,N_1-1} \frac{1}{h_{N_1-1}} N'_2(1)$	
$(4N_1, 4N_1)$	$\bar{\lambda}_{k-1,N_1-1} + 2\bar{\mu}_{k-1,N_1-1}$	

All other elements of matrix  $B_k^-$  are equal to zero.

The algorithm for computing of matrix  $B_k^+$  under the conditions (5) is described below.

1. Elements of the matrix  $B_k^+$  are determined by the formula

$$(B_k^+)_{p,q} = \delta_{p,q}, \quad p = 1, 2, \dots, 2N_1, \quad q = 1, 2, \dots, 2N_1, \quad (6)$$

where  $\delta_{p,q}$  is Kronecker delta.

It should be noted that computing of the elements of the matrix  $B_k^+$  by formula (6) corresponds to the boundary conditions (1)-(2).

2. The following values are assigned sequentially:  $i_g = 1, 2, \dots, N_1$ . For each fixed value of  $i_g$ , the actions listed below are performed.

2.1. The following values are assigned sequentially:  $i = 1, 2, \dots, N_1$ . For each fixed value of  $i$ , the actions listed below are performed.

2.1.1. The following values are assigned sequentially:  $j = 1, 2, \dots, N_{2,k}$ . For each fixed value of  $j$ , the actions listed below are performed.

2.1.1.1. The following values are assigned sequentially:  $q = 1, 2$ . For each fixed value of  $q$ , the actions listed below are performed.

2.1.1.1.1. Global index  $j_g$  is computed in accordance with formula

$$j_g = q + 2(i-1) + 2N_1(j-1). \quad (7)$$

2.1.1.1.2. As a vector of unknowns (see the formula (21) from the first part of this paper), we set

$$\bar{U}_k = \bar{e}_{j_g}, \quad (8)$$

where  $\bar{e}_{j_g}$  is the vector,  $j_g$ -th component of which is equal to 1, and the other components are equal to 0.

2.1.1.1.3. In accordance with the structure of vector of unknowns (8) with the use of corresponding formulas nodal components  $[\sigma_{1,2}^{(k)}]^{(i_g,1)}$ ,  $[\sigma_{2,2}^{(k)}]^{(i_g,1)}$  of stress tensor can be computed.

2.1.1.1.4. Elements of matrix  $B_k^+$  are computed in accordance with formulas

$$(B_k^+)_{2N_1+i_g j_g} = [\sigma_{1,2}^{(k)}]^{(i_g,1)}; \quad (B_k^+)_{3N_1+i_g j_g} = [\sigma_{2,2}^{(k)}]^{(i_g,1)}. \quad (9)$$

It should be noted that computing of the elements of the matrix  $B_k^+$  by formulas (9) corresponds to the boundary conditions (3)-(4).

### 3 Interface “discrete model – discrete-continual model”, “internal” boundary condition of the type “perfect contact”

Let's consider arbitrary boundary point  $x_{2,k}^b$ ,  $1 < k < n_b$  and corresponding interface between subdomain  $\omega_{k-1}^f$  (subdomain is approximated with the use of FEM,  $\rho_{k-1} = 1$ ) and subdomain  $\omega_k^{dc}$  (subdomain is approximated with the use of DCFEM,  $\rho_k = 2$ ).

Boundary conditions at section  $x_2 = x_{2,k}^b$  (perfect contact) have form ( $4N_1$  equations):

$$u_1^{(k-1,i,j)} = u_1^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1, \quad j = N_{2,k-1}; \quad (10)$$

$$u_2^{(k-1,i,j)} = u_2^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1, \quad j = N_{2,k-1}; \quad (11)$$

$$\sigma_{1,2}^{(k-1,i,j)} = \sigma_{1,2}^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1, \quad j = N_{2,k-1}; \quad (12)$$

$$\sigma_{2,2}^{(k-1,i,j)} = \sigma_{2,2}^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1, \quad j = N_{2,k-1}; \quad (13)$$

where  $\sigma_{1,2}^{(k,i)}(x_{2,k}^b + 0)$  and  $\sigma_{2,2}^{(k,i)}(x_{2,k}^b + 0)$  are nodal (node  $(k, i)$ ) values of functions of stress components  $\sigma_{1,2}(x_2)$  and  $\sigma_{2,2}(x_2)$  (after corresponding averaging) for discrete-continual finite element  $\omega_{k,i}^{dc}$ ;  $\sigma_{1,2}^{(k-1,i,j)}$  and  $\sigma_{2,2}^{(k-1,i,j)}$  are nodal (node  $(k, i, j)$ ) stress components  $\sigma_{1,2}$  and  $\sigma_{2,2}$  (after corresponding averaging).

Equations (1)-(4) can be rewritten in matrix form:

$$B_k^- \bar{U}_{k-1} = B_k^+ \bar{U}_k(x_{2,k}^b + 0), \quad (14)$$

where  $B_k^-$  is matrix of boundary conditions of size  $4N_1 \times 2N_1 N_{2,k-1}$ , which can be constructed in accordance with so-called method of basis variations [1-5,17];  $B_k^+$  is matrix of boundary conditions of size  $4N_1 \times 4N_1$ , which can be constructed in accordance with algorithm presented at Table 2.

**Table 2.** Algorithm of computing of nonzero elements of matrix  $B_k^+$ .

Numbers (indexes) of elements	Element value	Corresponding boundary condition
1	2	3
$(i, 2i-1), \quad i = 1, 2, \dots, N_1$	1	(10)
$(N_1 + i, 2i), \quad i = 1, 2, \dots, N_1$	1	(11)
$(2N_1 + 1, 2)$	$\bar{\mu}_{k,1} \frac{1}{h_1} N_1'(0)$	(12) $i = 1$
$(2N_1 + 1, 4)$	$\bar{\mu}_{k,1} \frac{1}{h_1} N_2'(0)$	
$(2N_1 + 1, 2N_1 + 1)$	$\bar{\mu}_{k,1}$	
$(2N_1 + i, 2(i-1)), \quad i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\mu}_{k,i-1} \frac{1}{h_{i-1}} N_1'(1)$	(12) $i = 2, 3, \dots, N_1 - 1$
$(2N_1 + i, 2i), \quad i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \left[ \frac{\bar{\mu}_{k,i-1}}{h_{i-1}} N_2'(1) + \frac{\bar{\mu}_{k,i}}{h_i} N_1'(0) \right]$	
$(2N_1 + i, 2(i+1)), \quad i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\mu}_{k,i} \frac{1}{h_i} N_2'(0)$	
$(2N_1 + i, 2(N_1 + i) - 1), \quad i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} (\bar{\mu}_{k,i-1} + \bar{\mu}_{k,i})$	

1	2	3
$(3N_1, 2N_1 - 2)$	$\bar{\mu}_{k,N_1-1} \frac{1}{h_{N_1-1}} N'_1(1)$	(12) $i = N_1$
$(3N_1, 2N_1)$	$\bar{\mu}_{k,N_1-1} \frac{1}{h_{N_1-1}} N'_2(1)$	
$(3N_1, 4N_1 - 1)$	$\bar{\mu}_{k,N_1-1}$	
$(3N_1 + 1, 1)$	$\bar{\lambda}_{k,1} \frac{1}{h_1} N'_1(0)$	(13) $i = 1$
$(3N_1 + 1, 3)$	$\bar{\lambda}_{k,1} \frac{1}{h_1} N'_2(0)$	
$(3N_1 + 1, 2N_1 + 1)$	$\bar{\lambda}_{k,1} + 2\bar{\mu}_{k,1}$	
$(3N_1 + i, 2i - 3),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\lambda}_{k,i-1} \frac{1}{h_{i-1}} N'_1(1)$	(13) $i = 2, 3, \dots,$ $N_1 - 1$
$(3N_1 + i, 2i - 1),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \left[ \frac{\bar{\lambda}_{k,i-1}}{h_{i-1}} N'_2(1) + \frac{\bar{\lambda}_{k,i}}{h_i} N'_1(0) \right]$	
$(3N_1 + i, 2i + 1),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} \bar{\lambda}_{k,i} \frac{1}{h_i} N'_2(0)$	
$(3N_1 + i, 2(N_1 + i)),$ $i = 2, 3, \dots, N_1 - 1$	$\frac{1}{2} (\bar{\lambda}_{k,i-1} + \bar{\lambda}_{k,i}) + \bar{\mu}_{k,i-1} - \bar{\mu}_{k,i}$	
$(4N_1, 2N_1 - 3)$	$\bar{\lambda}_{k,N_1-1} \frac{1}{h_{N_1-1}} N'_1(1)$	(13) $i = N_1$
$(4N_1, 2N_1 - 1)$	$\bar{\lambda}_{k,N_1-1} \frac{1}{h_{N_1-1}} N'_2(1)$	
$(4N_1, 4N_1)$	$\bar{\lambda}_{k,N_1-1} + 2\bar{\mu}_{k,N_1-1}$	

All other elements of matrix  $B_k^+$  are equal to zero.

The algorithm for computing of matrix  $B_k^-$  under the conditions (14) is described below.

1. Elements of the matrix  $B_k^-$  are determined by the formula

$$(B_k^-)_{p,q} = \delta_{p,q}, \quad p = 1, 2, \dots, 2N_1, \quad q = 1, 2, \dots, 2N_1. \quad (15)$$

It should be noted that computing of the elements of the matrix  $B_k^-$  by formula (15) corresponds to the boundary conditions (10)-(11).

2. The following values are assigned sequentially:  $i_g = 1, 2, \dots, N_1$ . For each fixed value of  $i_g$ , the actions listed below are performed.

2.1. The following values are assigned sequentially:  $i = 1, 2, \dots, N_1$ . For each fixed value of  $i$ , the actions listed below are performed.

2.1.1. The following values are assigned sequentially:  $j=1, 2, \dots, N_{2,k-1}$ . For each fixed value of  $j$ , the actions listed below are performed.

2.1.1.1. The following values are assigned sequentially:  $q=1, 2$ . For each fixed value of  $q$ , the actions listed below are performed.

2.1.1.1.1. Global index  $j_g$  is computed in accordance with formula

$$j_g = q + 2(i-1) + 2N_1(j-1). \quad (16)$$

2.1.1.1.2. As a vector of unknowns (see the formula (21) from the first part of this paper), we set

$$\bar{U}_{k-1} = \bar{e}_{j_g}. \quad (17)$$

2.1.1.1.3. In accordance with the structure of vector of unknowns (8) with the use of corresponding formulas nodal components  $[\sigma_{1,2}^{(k-1)}]^{(i_g, N_{2,k-1})}$ ,  $[\sigma_{2,2}^{(k-1)}]^{(i_g, N_{2,k-1})}$  of stress tensor can be computed.

2.1.1.1.4. Elements of matrix  $B_k^-$  are computed in accordance with formulas

$$(B_k^-)_{2N_1+i_g, j_g} = [\sigma_{1,2}^{(k-1)}]^{(i_g, N_{2,k-1})}; \quad (B_k^-)_{3N_1+i_g, j_g} = [\sigma_{2,2}^{(k-1)}]^{(i_g, N_{2,k-1})}. \quad (18)$$

It should be noted that computing of the elements of the matrix  $B_k^-$  by formulas (18) corresponds to the boundary conditions (12)-(13).

#### 4 Interface “discrete-continual model – discrete-continual model”, “internal” boundary condition of the type “perfect contact”

Let's consider arbitrary boundary point  $x_{2,k}^b$ ,  $1 < k < n_b$  and corresponding interface between subdomain  $\omega_{k-1}^{dc}$  (subdomain is approximated with the use of DCFEM,  $\rho_{k-1} = 2$ ) and subdomain  $\omega_k^{dc}$  (subdomain is approximated with the use of DCFEM,  $\rho_k = 2$ ).

Boundary conditions at section  $x_2 = x_{2,k}^b$  (perfect contact) have form ( $4N_1$  equations):

$$u_1^{(k-1,i)}(x_{2,k}^b - 0) = u_1^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1; \quad (19)$$

$$u_2^{(k-1,i)}(x_{2,k}^b - 0) = u_2^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1; \quad (20)$$

$$\sigma_{1,2}^{(k-1,i)}(x_{2,k}^b - 0) = \sigma_{1,2}^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1; \quad (21)$$

$$\sigma_{2,2}^{(k-1,i)}(x_{2,k}^b - 0) = \sigma_{2,2}^{(k,i)}(x_{2,k}^b + 0), \quad i = 1, 2, \dots, N_1. \quad (22)$$

Equations (19)-(22) can be rewritten in matrix form:

$$B_k^- \bar{U}_{k-1} (x_{2,k}^b - 0) = B_k^+ \bar{U}_k (x_{2,k}^b + 0); \quad (23)$$

where  $B_k^-$  and  $B_k^+$  are matrices of boundary conditions of size  $4N_1 \times 4N_1$ , which can be constructed in accordance with algorithms presented at Table 1 and Table 2 respectively.

## 5 Interface “discrete model – discrete model”, “internal” boundary condition of the type “perfect contact”

Let's consider arbitrary boundary point  $x_{2,k}^b$ ,  $1 < k < n_b$  and corresponding interface between subdomain  $\omega_{k-1}^{fe}$  (subdomain is approximated with the use of FEM,  $\rho_{k-1} = 1$ ) and subdomain  $\omega_k^{fe}$  (subdomain is approximated with the use of FEM,  $\rho_k = 1$ ).

It should be noted that in this case special formulation of boundary conditions are not required. Subdomains  $\omega_{k-1}^{fe}$  and  $\omega_k^{fe}$  are normally considered jointly within FEM, boundary conditions of the type “perfect contact” are automatically met.

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