Effects of Forces Applied in the Middle Plane on Bending of Medium-Thickness Band

Andrey Leontyev^{1,*}

¹Moscow state university of civil engineering, Yaroslavskoye shosse, 26, Moscow, Russia, 129337

Abstract. This paper examines the problem of calculating a plate of medium thickness uploaded by transverse loads and forces in the middle plane. We use variant equations from the theory of medium thickness plates as inputs, as proposed by B.F. Vlasov. The paper also considers the presence of tensile or compressive forces in the plane of the plate. The paper offers an example of calculation for a semi-infinite uploaded at the end of the transverse load and pivotally supported on the longitudinal edges. It is demonstrated that in the middle plane, the forces significantly influence the plate's deflections and bending forces. There are material differences between results obtained with the theory of plates of medium thickness and those of the classical the theory of bending of plates.

1 Introduction

A rectangular plate with various conditions on the contour, installed on an elastic base, is a widely used element in building structures. In certain cases, when calculating plates, particularly ones that rest on an elastic base, one needs to factor in some forces applied to the middle plane, in addition to the transverse loads. Such extra forces can be caused by seasonal and daily fluctuations of temperature, pre-tensed reinforcement steel, impacts from production equipment, and pressure of the enclosing walls on the foundation slab.

The stressed and deformed state of a medium thickness plate bending from the action of a transverse load, is described with these equations, as proposed by B.F. Vlasov [1]:

$$\nabla^2 \nabla^2 t(x, y) = \frac{p(x, y)}{D}, \quad \nabla^2 \varphi(x, y) - \frac{12}{h^2} \varphi(x, y) = 0 \quad (1)$$

Here, D is the cylindrical rigidity of the plate, h is its thickness. The deflection $w(x, y)$ and the rotation angles of the plate $t_x(x, y)$, $t_y(x, y)$ are associated with desired functions $t(x, y)$, $\varphi(x, y)$ as relations:

$$w = t - \frac{D}{Gh} \nabla^2 t, \quad t_x = \frac{\partial t}{\partial x} - \frac{D(1-\nu)}{2Gh} \frac{\partial \varphi}{\partial y}, \quad t_y = \frac{\partial t}{\partial y} + \frac{D(1-\nu)}{2Gh} \frac{\partial \varphi}{\partial x}, \quad (2)$$

where G – the shear modulus, ν – Poisson's ratio.

We can write the following dependence:

---

* Corresponding author: LeontievAN@mgsu.ru

$$\nabla^2 t(x, y) = \frac{\partial t_x(x, y)}{\partial x} + \frac{\partial t_y(x, y)}{\partial y}, \quad \varphi(x, y) = \frac{\partial t_y(x, y)}{\partial x} - \frac{\partial t_x(x, y)}{\partial y}. \quad (3)$$

$$M_x = -D \cdot \left(\frac{\partial^2 t}{\partial x^2} + \nu \frac{\partial^2 t}{\partial y^2} - \frac{D(1-\nu)^2}{2Gh} \frac{\partial^2 \varphi}{\partial x \partial y} \right), \quad (4)$$

$$M_y = -D \cdot \left(\frac{\partial^2 t}{\partial y^2} + \nu \frac{\partial^2 t}{\partial x^2} + \frac{D(1-\nu)^2}{2Gh} \frac{\partial^2 \varphi}{\partial x \partial y} \right),$$

$$M_{xy} = -D \frac{1-\nu}{2} \cdot \left[2 \frac{\partial^2 t}{\partial x \partial y} + \frac{D(1-\nu)}{2Gh} \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \right) \right], \quad (5)$$

$$Q_x = -D \cdot \left(\frac{\partial}{\partial x} \nabla^2 t - \frac{1-\nu}{2} \frac{\partial \varphi}{\partial y} \right), \quad Q_y = -D \cdot \left(\frac{\partial}{\partial y} \nabla^2 t + \frac{1-\nu}{2} \frac{\partial \varphi}{\partial x} \right). \quad (6)$$

2 State of the problem

If forces N_x , N_y , $N_{xy}=N_{yx}$ (Fig. 1) exist in the middle plane of the plate, the second equation (1) remains unchanged, and in the first equation must consider projections of such forces on axis Oz , created by the bending of the plate.

Fig. 1. Efforts in the middle plane of the plate

Notably, such forces are interrelated with equations of equilibrium:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_{yx}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \quad (7)$$

Project forces N_x , N_y , $N_{xy}=N_{yx}$ and their increments on the vertical axis. Then in the light of the expressions (2, 3, 7), the first equation (1) can be written as:

$$\nabla^2 \nabla^2 t - \frac{N_x}{D} \frac{\partial^2 t}{\partial x^2} - \frac{N_y}{D} \frac{\partial^2 t}{\partial y^2} - 2 \frac{N_{xy}}{D} \frac{\partial^2 t}{\partial x \partial y} = \frac{q}{D}. \quad (8)$$

3 Analytical method of solving

Let us assume that the longitudinal side edges of the plate have a hinged support (Fig. 2).

Fig. 2. Statement of the problem of calculation of the plate

Then our desired functions can be represented as:

$$t(x, y) = \sum_n W_n(x) \cos \beta_n y, \quad (9)$$

$$\varphi(x, y) = \sum_n F_n(x) \sin \beta_n y, \quad \text{where } \beta_n = \frac{n\pi}{b}. \quad (10)$$

We substitute decomposition (9) in equation (8), due to the orthogonality of trigonometric functions, for each of n get an ordinary differential equation of the fourth order, which in the case of only longitudinal forces N_x will appear as:

$$W_n^{IV} - 2\left(\beta_n^2 + \frac{N_x}{2D}\right)W_n'' + \beta_n^4 W_n = \frac{P_n}{D}, \quad (11)$$

and in the case of longitudinal forces N_y :

$$W_n^{IV} - 2\beta_n^2 W_n'' + \beta_n^4 \left(1 + \frac{N_y}{\beta_n^2 D}\right) W_n = \frac{P_n}{D}, \quad \text{where } P_n = \frac{2}{b} \int_0^b q(x, y) \cos \beta_n y dy. \quad (12)$$

Equations (11) and (12) can be represented using the standard form, typical for a problem of the bending of the beam located on an elastic base:

$$W_n^{IV} - 2r_n^2 W_n'' + s_n^4 W_n = \frac{P_n}{D}. \quad (13)$$

Depending on the relation of the coefficient values r_n and s_n we find the type of roots of the respective characteristic equations, and therefore the type of particular integrals for its solution.

First examine a case of the plate under the action of forces N_y , distributed evenly over its longitudinal edges (Fig. 2). By comparing equations (12) and (13) we can see that in this case, the coefficients of equation (13) will be values:

$$r_n^2 = \beta_n^2, \quad s_n^4 = \beta_n^4 \left(1 + \frac{N_y}{\beta_n^2 D}\right), \quad (14)$$

relations between which depend on the sign of forces N_y : in the case of stretching plate ($N_y > 0$) $s_n > r_n$, in the case of compression ($N_y < 0$) $s_n < r_n$.

For $s_n > r_n$ the roots k_n of the characteristic equation is determined by the formula:

$$k_n = \pm \alpha_n \pm i\gamma_n, \quad (15)$$

where

$$\alpha_n = \sqrt{\frac{s_n^2 + r_n^2}{2}}, \quad \gamma_n = \sqrt{\frac{s_n^2 - r_n^2}{2}}, \quad (16)$$

while the general solution of differential equation (11) is written as:

$$W_n(x) = C_{1n}F_{1n}(x) + C_{2n}F_{2n}(x) + C_{3n}F_{3n}(x) + C_{4n}F_{4n}(x) + W_n^0(x). \quad (17)$$

Here, $F_{1n}(x), \dots, F_{4n}(x)$ are hyperbolic-trigonometric functions, $W_n^0(x)$ is a particular solution that depends on the transverse load $p_n(x)$.

For $s_n < r_n$ the roots of the characteristic equation will appear as:

$$k_1 = -k_2 = \lambda_{1n} = \sqrt{r_n^2 + \sqrt{r_n^4 - s_n^4}}, \quad k_3 = -k_4 = \lambda_{2n} = \sqrt{r_n^2 - \sqrt{r_n^4 - s_n^4}}, \quad (18)$$

and therefore the solution to equation (11) will have this form:

$$W_n(x) = C_{1n} \text{sh} \lambda_{1n} x + C_{2n} \text{ch} \lambda_{1n} x + C_{3n} \text{sh} \lambda_{2n} x + C_{4n} \text{ch} \lambda_{2n} x + W_n^0(x). \quad (19)$$

If the plate is under the action of forces N_x coefficients of equation (13) appear:

$$r_n^2 = \beta_n^2 \left(1 + \frac{N_x}{2\beta_n^2 D} \right), \quad s_n^4 = \beta_n^4. \quad (20)$$

From the relations (20) we can see that this time, in the case of stretching plate ($N_x > 0$) $s_n < r_n$, and in the case of compression plate ($N_x < 0$) $s_n > r_n$, the solutions to the problem will be represented respectively as expressions (19) or (17).

Us now examine the problem of the bending of a semi-infinite plate under the action of transverse load $q(y)$ in its initial section and longitudinal compressing forces N_y , evenly distributed over its longitudinal edges (Fig. 2).

In this case, if the transverse load is represented as $q_y = q_0 \cos \beta y$, where $\beta = \pi/b$, then even first term of decomposition (9) present an accurate solution, and so index n can be omitted everywhere. Assuming the plate's infinite length, solution to equation (13) can be written as:

$$W(x) = C_1 e^{-\lambda_1 x} + C_2 e^{-\lambda_2 x}. \quad (21)$$

To completely solve the problem of the bending of a medium-thickness plate, we also need to add to expression (19) the integral of the second equation from (1), which in the light of decomposition of (10) and the infinite length of the plate, can be written as:

$$F(x) = D_1 e^{-gx}, \quad \text{where } g = \sqrt{\beta^2 + \frac{Gh}{D(1-\nu)}} = \frac{1}{h} \sqrt{12 + \beta^2 h^2}. \quad (22)$$

To find the three integration constants C_1 , C_2 , D_1 at the initial section ($x = 0$), one can set the following boundary conditions:

$$\begin{aligned} M_x &= -D \left[W'' - \nu \beta^2 W - \frac{h^2(1-\nu)}{12} \beta F' \right] \cos \beta y = 0, \\ M_{xy} &= -D(1-\nu) \left[-\beta W' + \frac{h^2}{24} (F'' + \beta^2 F) \right] \sin \beta y = 0, \\ Q_x &= -D \left[W''' - \beta^2 W' - \frac{(1-\nu)}{12} \beta F \right] \cos \beta y = -q \cos \beta y. \end{aligned} \quad (23)$$

Revealing these conditions using equations (21) and (22), we obtain a system of three algebraic equations:

$$\begin{aligned} (\lambda_1^2 - \nu \beta^2) C_1 + (\lambda_2^2 - \nu \beta^2) C_2 + \frac{h^2(1-\nu)}{12} \beta g D_1 &= 0, \\ \beta (\lambda_1 C_1 + \lambda_2 C_2) + \frac{h^2}{24} (\beta^2 + g^2) D_1 &= 0, \\ \lambda_1 (-\lambda_1^2 + \beta^2) C_1 + \lambda_2 (-\lambda_2^2 + \beta^2) C_2 - \frac{1-\nu}{2} \beta D_1 &= \frac{q_0}{D}. \end{aligned} \quad (24)$$

Consider the same problem with existing tensile forces of N_y , the solution to (17) can be written as:

$$W(x) = C_1 e^{-\alpha x} \sin \gamma x + C_2 e^{-\alpha x} \cos \gamma x. \quad (25)$$

To find the constants of integration, boundary conditions (23) and expressions (22) and (25) allow to obtain the following equation:

$$\begin{aligned} -2\alpha\gamma C_1 + (\alpha^2 - \gamma^2 - \nu\beta^2)C_2 + \frac{h^2(1-\nu)}{12}\beta g D_1 &= 0, \\ -\beta\gamma C_1 + \beta\alpha C_2 + \frac{h^2}{24}(g^2 + \beta^2)D_1 &= 0, \end{aligned} \quad (26)$$

$$\gamma(3\alpha^2 - \gamma^2 - \beta^2)C_1 + \alpha(3\gamma^2 - \alpha^2 + \beta^2)C_2 - \frac{1-\nu}{2}\beta D_1 = \frac{q_0}{D}.$$

Having solved the equation systems (24) and (26) and defining the constants C_1 , C_2 , D_1 , we can determine our calculated values using the formulas of the theory of plate middle-thickness [1]. For example, the deflection $w(x,y)$ and the bending moment $M_y(x,y)$ are determine as:

$$w(x,y) = \left[W(x) - \frac{12}{h^2} (W''(x) - \beta^2 W(x)) \right] \cos \beta y, \quad (27)$$

$$M_y(x,y) = -D \left[-\beta^2 W(x) + \nu W''(x) - \frac{h^2(1-\nu)}{12} \beta F'(x) \right] \cos \beta y, \quad (28)$$

where function $W(x)$ that is part of (27) and (28) appears as (21) when the plate is compressed, and as (25) when the plate is stretched.

4 Example of calculation

Fig. 3 has graphs representing the dependency of dimensionless deflection $\overline{w_0} = w(0,0)D/p_0 b^3$ in the middle of the plate's initial section on dimensionless longitudinal forces $\overline{N}_y = N_y b^2 / D$.

Fig. 3. Diagrams of dependence of deflection on longitudinal forces

In the diagram, solid lines 1 and 2 relate respectively to the cases compressed and stretched of the plate middle-thickness for $h/b = 0.2$, while dotted lines 3 and 4 describe the cases for the thin compressed and stretched plate.

Fig. 4 shows dependencies of dimensionless bending moments $\bar{M}_y^0$ at the middle of the initial section, on dimensionless longitudinal forces $\bar{N}_y$.

Fig. 4. Diagrams of dependence of the bending moments on longitudinal forces

5 Conclusions

Can see that an increase in longitudinal force N_y causes material changes of deflection and bending moments: they are higher in the case of compression, and lower in the case of stretching. Expressed as percentage, deflections vary more than bending moments. It should also be noted that the effect of longitudinal forces on the plate middle-thickness is somewhat stronger than on a thin plate, and it is greater with compression than with stretching. Lines 1 and 3, nearly straight at low values of N_y , become curves tending to rise abruptly as N_y increases, approaching their respective critical values.

References

1. B.F. Vlasov, Equation of bending plates of average thickness. / Theoretical and experimental studies of strength and stiffness of structural elements, Moscow, MSUCE, (1989)
2. R.F. Gabbasov, V.V. Filatov, Calculation of compressed-bent plates in incomplete contact with an elastic Foundation, Moscow, MSUCE, (1999)
3. S.P. Timoshenko, S. Voynovskiy-Kriger. / Plates and shells, Moscow, Nauka (1966)
4. A.T. Turganbaev, Bending of a rectangular plate located on an elastic Winkler's Foundation with the influence of the longitudinal force. / Bases, foundations and soil mechanics, **3** (1993)