# Stress relaxation of constructions elements

*Evgeniy* Larionov[1],*, *Evgeniy* Zveryaev [2]

[1]Moscow state university of civil engineering, Yaroslavskoye shosse, 26, Moscow, Russia, 129337
[2]Keldysh Institute of Applied Mathematics, Miusskaya sq. 4, Moscow, 125047, Russia

**Abstract.** A relaxation of stress in the elements of constructions is considered and an approach is proposed for solution of corresponding problems. It is notable that this approach is based on a modification Boltzmann's principle superposition of fraction creep deformations. This modification reduces the noted problems to solution of linear relative to so-called structural stress integral equations. Next a desired stress is defined by solution of algebraic equations. It should be underline that a material (concrete, steel, graph) of elements a considered as a union of its fractions with statistical disturbed strengths. This ascending to Weibull conception [1] permits to modify Boltzmann's principle superposition [2]. As a result this principle is applicable when a dependence on deformations from the stresses is nonlinear [3-7].

According to Berg [6] and Gvozdev [9] an increasing on cross-section loading $N(\tau)$ implies destruction a certain fractions of the element. Here $\tau$ is a current temp.
In consequence of this destruction an initial area $A$ of cross-section is decreased until $A(\tau)<A$. The area $A(\tau)$ represents a cross-section of all entire at instant $\tau$ fractions of the constructive element. $A$ connected with the structural damages of the element value

$$\sigma_c(\tau)=\frac{N(\tau)}{A(\tau)} \quad (1)$$

is called the structural stress. Since

$$\sigma_c(\tau)=\frac{N(\tau)}{A} \quad (2)$$

is the normal stress then according to the relations (1) and (2) we have

$$\sigma_c(\tau)=\frac{A}{A(\tau)}\sigma(\tau)=S^0(\tau)\sigma(t) \quad (3)$$

The function $S^0(\tau)$ describs the destruction of a certain part of element and is called the function of non-linearity of stress.

As $\sigma(\tau)=const$ ; $\tau\in[t_0,t]$ we obtain the creep deformation

* Corresponding author: larionov.ea@gmail.com

$$\varepsilon_c\left(t_0,t\right)=C_0\left(t,t\right)\sigma\left(t\right) \tag{4}$$

and the complete deformation is

$$\varepsilon\left(t,t_0\right)=\frac{\sigma\left(t\right)}{E\left(t\right)}+C_0\left(t,t_0\right)\sigma\left(t\right) \tag{5}$$

Here $\frac{\sigma\left(t\right)}{E\left(t\right)}$ is the instantaneous deformation $\varepsilon_m\left(t\right)$; $C_0\left(t,t_0\right)$ is the measure of the simple creep; $E\left(t\right)$ is the elasticity module.

The increment $\Delta\sigma\left(t\right)=\sigma\left(t\right)-\sigma\left(t_0\right)$ of stress implies the creep deformation $\Delta\varepsilon_c\left(t,t_0\right)$. In virtue of the Boltzmann's principle of superposition the fraction increment $\Delta\varepsilon_c\left(t,\tau_i\right)$ is defined only by the value and duration $\Delta\sigma\left(\tau_i\right)$, and is independent on the rest increments $\Delta\sigma\left(\tau_j\right)$ ; $j\neq i$ .

Consequently

$$\Delta\varepsilon_c\left(t,\tau_i\right)=C_0\left(t,\tau_i\right)\Delta\sigma\left(\tau_i\right) \tag{6}$$

$$\Delta\varepsilon_c\left(t,t_0\right)=\sum_{i=1}^{n}C_0\left(t,\tau_i\right)\Delta\sigma\left(\tau_i\right) \tag{7}$$

$$\Delta\varepsilon_c\left(t,t_0\right)=\int_{t_0}^{t}C_0\left(t,\tau\right)d\sigma\left(\tau\right) \tag{8}$$

and integrating by parts, we have

$$\Delta\varepsilon_c\left(t,t_0\right)=C_0\left(t,t\ \right)\sigma\left(t\right)-C_0\left(t,t_0\right)\sigma\left(t_0\right)-\int_{t_0}^{t}\sigma\left(\tau\right)\frac{\partial C_0\left(t,\tau\right)}{\partial\tau}dt \tag{9}$$

Then, according to the equalities

$$\varepsilon\left(t,t_0\right)=\frac{\sigma\left(t\right)}{E\left(t\right)}+\varepsilon_c\left(t,t_0\right) \tag{10}$$

$$\varepsilon_c\left(t,t_0\right)=C_0\left(t,t_0\right)\sigma\left(t_0\right)+\Delta\varepsilon_c\left(t,t_0\right) \tag{11}$$

$$C_0\left(t,\tau\right)=\frac{1}{E\left(\tau\right)}-\frac{1}{E\left(t\right)}+C\left(t,\tau\right) \tag{12}$$

we obtain the linear rheological state equation

$$\varepsilon\left(t,t_0\right)=\frac{\sigma\left(t\right)}{E\left(t\right)}+C\left(t,t\right)\sigma\left(t\right)-\int_{t_0}^{t}\sigma\left(\tau\right)\frac{\partial}{\partial\tau}\frac{1}{E\left(\tau\right)}d\tau-\int_{t_0}^{t}\sigma\left(\tau\right)\frac{\partial C\left(t,\tau\right)}{\partial\tau}d\tau \tag{13}$$

In the equation (13) the value $C\left(t,t\right)\sigma\left(t\right)$ represents the so-called quick creep deformation [10].

In consequence of the destruction of a certain fractions of the element and the next redistribution of the loading $N\left(\tau\right)$ every increment $\Delta\sigma\left(\tau_k\right)$ intensifies the action of the previous increment $\Delta\sigma\left(\tau_i\right);\left(i<k\right)$. This means the dependence of fraction creep deformation $\Delta\varepsilon_c\left(t,\tau_i\right)$ also on all next increments of the stress $\sigma\left(\tau\right)$ and therefore the superposition of $\Delta\varepsilon_c\left(t,\tau_i\right)$ relatively $\Delta\sigma\left(\tau_i\right)$ is impossible.

Non we show that this superposition is realized relatively $\Delta\sigma_c\left(\tau_i\right)=S^0\left(\tau_i\right)\Delta\sigma\left(\tau_i\right)$, of the fraction increments of the structural stress $\sigma_c\left(\tau\right)$ .

An interdependence on $\varepsilon_c(t,\tau_i)$ take place for an ideal element when all the fractions have the same strengths $R(\tau)$ and consequently the retribution of $N(\tau)$ is excluded. Therefore parallel with the given element we consider the geometrically identical to it ideal element with the same parameters $E(\tau)$ and $C(t,\tau)$. The ideal element under the axial loading $N_c(\tau)=\frac{A}{A(\tau)}N(\tau)=S^0(\tau)N(\tau)$ tests the stress $\sigma_c(\tau)$ and the increment $\Delta\sigma_c(\tau_i)$ generate $\Delta\varepsilon_c(t,\tau_i)$.

Thus we have

$$\Delta\varepsilon_c(t,\tau_i)=C_0(t,\tau_i)\Delta\sigma_c(\tau_i) \tag{14}$$

$$\Delta\varepsilon_c(t,t_0)=\sum_{i=1}^{n}C_0(t,\tau_i)\Delta\sigma_c(\tau_i) \tag{15}$$

$$\Delta\varepsilon_c(t,t_0)=\int_{t_0}^{t}C_0(t,\tau)d\sigma_c(\tau) \tag{16}$$

$$\Delta\varepsilon_c(t,t_0)=C(t,t)\sigma_c(t)-C(t,t_0)\sigma_c(t)-\int_{t_0}^{t}\sigma_c(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E_{(\tau)}}+C(t,\tau)\right]d\tau \tag{17}$$

$$\varepsilon_c(t,t_0)=C(t,t)\sigma_c(t)-\int_{t_0}^{t}\sigma_c(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E_{(\tau)}}+C(t,\tau)\right]d\tau \tag{18}$$

According to equation (18) and $\sigma_c(\tau)=S^0(\tau)\sigma(\tau)$ we obtain the nonlinear rheological state equation

$$\varepsilon(t,t_0)=S^0(t)\sigma(t)\left[\frac{1}{E(t)}+C(t,t)-\int_{t_0}^{t}\frac{S^0(\tau)\sigma(\tau)}{S^0(t)\sigma(t)}\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)}+C(t,\tau)\right]d\tau\right. \tag{19}$$

We can present an alternate vision of the deduction for nonlinear rheological equation (19). Consider to this end the part $V_t$ of the constructive element consisting from the union of all the remaining entire in the interval $[t_0,t]$ fractions. The loading $N(\tau)$ generates on its cross-section (with the area $A(t)$) same structural stress $\sigma_c(\tau)$ that on cross-section of the considered element (with the area $A(\tau)$).

The element and its part $V_t$ have same fraction creep deformations and the absence of one in the interval $[t_0,t]$ of the redistribution of the loading $N_e(\tau)=\frac{A(t)}{A}N(\tau)$ implies their interindepency. Then, for $V_t$ we have the relations (14)-(18) and consequently again obtain the rheological equation (19).

Evidently linear equation (13) contradicts to the knowns non-linear diagrams «$\sigma-\varepsilon$».

Note that according to A.A. Gvozdev a non-linear part of «$\sigma-\varepsilon$» is generated by quick creep at the momentum of loading [19]. In addition he assumes the linearity of the dependence $\varepsilon_m(\tau)$ on $\sigma(\tau)$. By A.A. Gvozdev the creep deformation is the composition of the linear component $\varepsilon_{ce}(t,t_0)=\int_{t_0}^{t}C^*(t,\tau)d\sigma(\tau)$ and the non-linear component

$\varepsilon_{cn}(t,t_0)=\int_{t_0}^{t}L(t,\tau,\sigma)d\sigma(\tau)$, generated because of the structural damages. This conception implies a necessity of the choice of the functions $C^*$ and $L$ corresponding to the diagram «$\sigma-\varepsilon$». We note that this case reduce to the considerable inconvenience for the applications [12].

In connection with diagram «$\sigma-\varepsilon$» V.M. Bondarenko affirms that the dependence $\varepsilon_m(t)$ from $\sigma(t)$ also is non-linear and is described by the relation $\varepsilon_m(t)=S_m[\sigma(t)]/E(t)$ [13]. Since the instantaneous and creep deformations generated by the same stress $\sigma_c(\tau)$ their non-linear functions $S_m[\sigma(\tau)]$ and $S_c[\sigma(\tau)]$ coincide $S_m[\sigma(\tau)]=S_c[\sigma(\tau)]=A/A(\tau)$ [3, 4].

In [14–16] the non-linear rheological equation is presented in the following form

$$\varepsilon(t)=\varepsilon_m(t)-\int_{t_0}^{t}\sigma(\tau)\frac{\partial C(t,\tau)}{\partial\tau}d\tau \tag{20}$$

This means that it is assumed (along with $C(t,t)=0$ and $E(\tau)=const$) the possibility of the superposition $\Delta\varepsilon_c(t,\tau_i)$ relative to $\Delta\sigma(\tau_i)$. However (as is shown in section 2) such superposition takes place only in the linear creep theory.

**Remark 1.** The adequate to the physical essence of the creep phenomenon the non-linear rheological equations are deduced only on basis of the corresponding modification Boltzmann's principle superposition.

Papiers [14–16] are useful for an analysis of the applied in our actual norms creep theory. In addition they are connected with the problem of correspondence of these norms and Eurocod.

**Remark 2.** A creep phenomenon (increment of $\varepsilon(t)$ under $\sigma(\tau)=\sigma(t_0)$; $\tau\in[t_0,t]$) is the consequence of strength distribution on cross-section of the element. In fact a degradation of $R(\tau)$ implies an increment $\sigma_c(\tau)=\dfrac{A}{A(\tau)}\sigma(t_0)$ and therefore an increment of $\varepsilon(t)$.

**Remark 3**. In the course of taking down of the loading we have $S^0[\sigma(\tau)]=A/A(\tau)=const$, hence, the dependence $\varepsilon(\tau)$ on $\sigma(\tau)$ is linear, that is observed on diagram «$\sigma-\varepsilon$».

According to the equation (19) we have

$$\varepsilon(t,t_0)=\frac{\sigma_c(t)}{E(t)}+C(t,t)\sigma_c(t)-\int_{t_0}^{t}\sigma_c(\tau)\frac{\partial C_0}{\partial\tau}(t,\tau)d\tau \tag{21}$$

$$\sigma_c(t)=\hat{\sigma}_c(t)+\lambda(t)\int_{t_0}^{t}\sigma_c(\tau)\frac{\partial C_0(t,\tau)}{\partial\tau}d\tau \tag{22}$$

Here $\hat{\sigma}(t)=\dfrac{\varepsilon(t,t_0)E(t)}{1+E(t)C(t,t)}$; $\lambda(t)=\dfrac{E(t)}{1+E(t)C(t,t)}$ is the elastoplastic module of deformations. Solving the linear integral in equation (22) (par example by the iteration method) we come to an equation $S^0[\sigma(t)]=\sigma_c^*(t)$, where $\sigma_c^*(t)$ is the solution of equation (22).

In applications the function $S^0(\tau)$ is represented usually in form

$$S^0\left[\sigma(t)\right]=1+V\cdot\left[\frac{\sigma(t)}{R(t)}\right]^m \text{ or } S^0[\sigma(t)]=a[\sigma(t)]^b$$

where $V$ , $m, a, b$ are the experimental parameters [17]. Then, we find $\sigma(t)$ as a solution of equation

$$\left[\sigma(t)\right]^{m+1}+\frac{\left[R(t)\right]^m}{V}\sigma(t)-\frac{\left[R(t)\right]^m}{V}\sigma_c^*(t)=0 \text{ or } a\left[\sigma(\tau)\right]^{b+1}=\sigma_c^*(t)$$

If $C(t,\tau)=C(0,\infty)\left[1-\beta e^{-\gamma(t-\tau)}\right]$ the stress $\sigma(t)$ is defined by solution of corresponding to (19) linear differential eqnation.

As the loading $N(\tau)=N(t_0)$ at $\tau\geq t_0$ we have $S^0\left[\sigma(t)\right]=S^0\sigma(t_0)$ and therefore $\sigma(t)=\sigma_c^*(t)/S^0\left[\sigma(t_0)\right]$. This case take place when $\varepsilon(\tau,t_0)=\varepsilon(t_0)$ at $\tau\geq t_0$.

Let us illustrate the simple example of the stress relaxation in the concrete. Suppose that a reinforced concrete rod tests with the axial loading

$$N(\tau)=\begin{cases}N;2kt\leq\tau<(2k+1)T\\N_0;(2k+1)T\leq\tau<2(k+1)T\end{cases} \tag{23}$$

where $T>0$; $N_0<N$; $k=0,1,2,......,n$ .

The effort $N$ on the concrete part of the cross-section and $N_0$ generates the stresses

$$\sigma_1=\frac{N}{A_c(1+m\mu)} \text{ and } \sigma_2=\frac{N_0}{A_c(1+m\mu)} \tag{24}$$

at the moment $t_1=2kt$ and $t_2=(2k+1)T$ respectively. Here $m=E_s/E_c$; $\mu=A_s/A_c$; $E_c(E_s)$ is the elasticity module of the concrete (steel); $A_c(A_s)$ is the area of the cross-section of the concrete (steel).

Let $A_1$ be the corresponding to $\sigma_1$ area of the cross-section for all entire fractions. Taking into account that $A(\tau)=const$ on increasing loading and $\sigma_2<\sigma_1$ we obtain

$$S^0(\tau)=1+V\cdot\left[\frac{\sigma_1}{R}\right]^m;\tau\geq 0 \tag{25}$$

In given by relation (23) the intervals of temps we have

$$\sigma(t)=\sigma_i+\lambda\int_{t_i}^t\sigma(\tau)e^{-\gamma(t-\tau)}d\tau; i=1,2; \tag{26}$$

Here

$$\lambda=\frac{E_s\gamma\mu\beta C_o}{1+m\mu}; t_1=2kt;\ t_2=(2k+1)T; k=0,1,2,......., n$$

The equations is reduced to the linear differential ones

$$\frac{d\sigma(t)}{dt}+\delta\sigma(t)=\sigma_i\gamma; \delta=\gamma+\lambda; \quad i=1,2 \tag{27}$$

The solution of the equation (27) is represented by the following function

$$\sigma(t)=\sigma_i\left[\frac{\gamma}{\delta}+\frac{\lambda e^{-\delta(t-nT)}}{\delta}\right] \tag{28}$$

where

$$\sigma_i = \begin{cases} \sigma_1; & n = 2p \\ \sigma_2; & n = 2p+1 \end{cases} \tag{29}$$

In [12] the noted problems are decided by the small parameter method. The stress $\sigma(t)$ is looked in the form

$$\sigma(t) = \sum_{i=0}^{\infty} \xi^i \sigma_i(t) \tag{30}$$

where $\xi$ is the small parameter and $\sigma(t)$ is the solution of corresponding linear integral equation. Evidently this approach reduces to a considerable calculs in applications.

## References

1. W. Weibull, Trans. Roy. Inst. Technol.(Stockholm), 27 (1949)
2. L. Boltzmann, [WA-1, 616] (Wien. Ber. 70. S. 275-306. (1874) und Pogg. Ann. Erg.-Bd. 7. S. 624 (1876))
3. V.M. Bondarenko, E.A. Larionov, Str. Mech. of Eng. Constr. and Build, 2 (2011)
4. E.A. Larionov, A.E. Larionov. Str. Mech. and Calculi build, 2 (2015)
5. N.T. Vasilkova, M.E. Bashcatova, E.A. Larionov, Str. Mech. of Eng. Constr. and Build,1 (2012)
6. E.A. Larionov, Vestnik MGSU, 4 (2016)
7. E.A. Larionov, Str. Mech. of Eng. Constr. and Build, 4 (2015)
8. O. Berg. *The physical bases of strength theorg of concrete and reinforced concrete.* (Stroyizdat, 1962)
9. A.A. Gvozdev. *A calcul of bearing capacity of constructions by limit equilibrium method*. (Stroyizdat, 1949)
10. Yu. N. Rabotnov. *A creep of constructive elements*, (Nauka, 1982)
11. A.A. Gvozdev, Mech. Solid, 5 (1972)
12. K.Z. Galustov. *Non-linear creep theory of Concrete and calculation of reinforced constructions*. (Fizmatlit, 2006)
13. V.M. Bondarenko, S.V. Bondarenko, *Ingeneering methods of non-linear theory of reinforced concrete*. (Stroyizdat, 1982)
14. R.S. Sanzharovskij. Str. Mech. of Eng. Constr. 1 (2014)
15. R.S. Sanzharovskij, M.M. Manchenko, Str. Mech. and Calculi build, 2 (2015)
16. R.S. Sanzharovskij, M.M. Manchenko, Str. Mech. of Eng. Constr. 3 (2016)
17. V.M. Bondarenko, Str. Mech. of Eng. Constr. 2 (2015)