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Modal approach in structural-dynamic equations

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Abstract. Some aspects of the solution of inhomogeneous problems of rods and plates are considered in the article. Dynamic effects from the effects of short-term loads are revealed with the help of modal decomposition under the assumption of the constancy of the elastic properties of concrete corresponding to its age during the application of the perturbing dynamic effect. Superposition of reactions to long-and-short-term effects are possible due to the inadmissibility of finite deformations in building structures.

1 Introduction

For the solution of the dynamic problems known averaging method [1, 2, 3], where there is a limitation: for the implementation of the method requires that the material have low relaxation or the step of falling stress is a "small parameter" in the experience of relaxation. The method of expansion in eigenfunctions of a linear homogeneous operator with constant coefficients has no such restrictions, which form the system of orthogonal functions [4], satisfying the conditions of theorem Steklov [4, 5, 6], i.e. it is possible to build an absolutely convergent generalized Fourier series [7]. As such a system, one can effectively use the solution of the problem of free vibrations of linearly elastic bodies with homogeneous kinematic conditions [8, 9]. In the literature the forms of free oscillations is often called fundamental mode oscillation, then this method is called modal approach. In structural mechanics there are many analytical solutions of problems of free oscillations of rods and plates. The based on them solutions of the inhomogeneous problems (forced oscillations, vibrations, impacts with different pulse shape) you can also get analytical. Further development of the proposed theory envisages the construction of analytical expressions for reinforced concrete structural elements.

2 Evolutionary model of hereditary-viscoelastic material

The most suitable for describing the rheological properties of the material should be considered viscoelastic the models hereditary ageing [10]. For such models, an analytical

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expression has the form:

$$\sigma(t) = E(t)\varepsilon(t) - \int_{\tau_0}^{t} \sigma(\tau)K(t,\tau)d\tau \quad (1)$$

where $K(t,\tau)$ – creep memory function, $\sigma(\tau)$ – development of stress, $\varepsilon(t)$ – deformation, $E(t)$ – modulus of elasticity of instantaneous deformation.

Thus, in order to find the stress, it is necessary to solve the integral equation (1).

Using the results of [1], one can take the relations for relaxation in the form:

$$\sigma(t) = E(t)\varepsilon(t) - \int_0^t \Gamma(t,\tau)\varepsilon(\tau)d\tau = E(t)\left[\varepsilon(t) - \int_0^t T(t-\tau)\varepsilon(\tau)d\tau\right] \quad (2)$$

where t – moment of observation; $E(t)$ – modulus of elasticity, which depends on time; $\varepsilon(t)$ – deformation, small, i.e. $\varepsilon >> \varepsilon^2$; $\sigma(t)$ – Conditional stress; $\Gamma(t,\tau)$ – gamma function or dimensional kernel of relaxation; $T(t-\tau)$ – a positive function belonging to the space of L1 functions [6] or space of integrable functions on the interval $(0, \infty)$, i.e.

$$\int_0^\infty T(\tau)d\tau = C < \infty \quad (3)$$

The latter property can be enhanced, bearing in mind of equilibrium stress that in the experiment on the relaxation is not zero:

$$\int_0^\infty T(\tau)d\tau = C < 1 \quad (4)$$

In contrast to [1] modulus elasticity in (2) is assumed to be independent of the moment of observation. Dependence of modulus elasticity on time, following [9], written in the form:

$$E(t) = E_0\left(1 - \beta e^{\alpha t}\right) \quad (5)$$

For the parameters of this ratio, there is a clear physical meaning: E_0 – "equilibrium" module that corresponds to the ending of the aging process (при $t \to \infty$); β – is the relative change of modulus with age t, where the parameter α determines the speed of aging:

$$\beta = \frac{E_0 - E_\infty}{E_0}$$

In [11, table 4] given the data to determine the parameters of the formula (5).

3 Modal approach

Consider a viscoelastic body whose properties are described by expression $\sigma(t) = E(t)\varepsilon(t) - \int_0^t \Gamma(t,\tau)\varepsilon(\tau)d\tau = E(t)\left[\varepsilon(t) - \int_0^t T(t-\tau)\varepsilon(\tau)d\tau\right]$ (2), written in the form:

$$\hat{\sigma}(t) = \hat{\hat{\mathrm{E}}} \cdot\cdot \varepsilon(t) - \int_0^t \hat{\hat{\mathrm{T}}}_{dyn}(t-\tau) \cdot\cdot \hat{\varepsilon}(\tau)d\tau \quad (6)$$

Here $\hat{\sigma}$, $\hat{\varepsilon}$ – tensors of rank II of the stresses and small deformations; $\hat{\hat{\mathrm{E}}}$, $\hat{\hat{\mathrm{T}}}$ – tensors of rank IV instantaneous modulus of elasticity and of relaxation memory function.

In the General case the tensor of elastic properties $\hat{\hat{\mathrm{E}}}$ is considered constant by in the equation (5), moreover, the start timing is the moment occurrence of dynamic programming process, a but age of concrete has its start timing; the tensor of kernels of relaxation is calculated at a certain age of the concrete from wave or other dynamic experiments. To determine the dynamic states of an object, we use the Lagrange-d'Alembert variational equation [12]:

$$\int_V \delta\hat{\varepsilon}(t)\cdot\cdot\left[\hat{\hat{\mathrm{E}}}\cdot\cdot\hat{\varepsilon}(t)-\int_0^t \hat{\hat{\mathrm{T}}}_{dyn}(t-\tau)\cdot\cdot\hat{\varepsilon}(\tau)d\tau\right]dV+\int_V \delta\mathbf{u}(t)\cdot\rho_0\ddot{\mathbf{u}}(t)dV- \\ -\int_V \delta\mathbf{u}\cdot\rho_0\mathbf{k}dV-\int_S \delta\mathbf{u}\cdot\mathbf{f}_0 dS=0 \tag{7}$$

Here ρ_0 – density; **u** – вектор перемещений; **k, f**$_0$ – the vectors of mass and surface loads, in the adopted configuration; δ – denotes kinematically admissible variation; character (·) denotes tensor convolution; the integrals are computed on the undeformed volume and surface of the body. Acceleration of the material point is identified with the second partial time derivative. It is assumed that the properties of the material and the kinematic boundary conditions are unchanged during the deformation process.

Suppose that an analytic solution of the problem of free vibrations of a linearly elastic body is known (so in (7) we must eliminate the summand, which responsible for the viscoelasticity), i.e. known countable set the frequencies of free vibration ω_k and countable set of modes of vibration $\mathbf{Y}_k(\mathbf{R})$ (k=1, 2, 3…N), $N<\infty$; $\boldsymbol{R}$ – the radius-vector of the current point. These forms have properties:

$$\int_V \rho_0\mathbf{Y}_k(\mathbf{R})\cdot\mathbf{Y}_m(\mathbf{R})dV=\begin{cases}1\forall k=m\\ 0\forall\neq m\end{cases} \\ \int_V def(\mathbf{Y}_k(\mathbf{R}))\cdot\cdot\hat{\hat{\mathrm{E}}}\cdot def(\mathbf{Y}_m(\mathbf{R}))dV=\begin{cases}\omega_{0n}^2\forall k=m\\ 0\forall\neq m\end{cases} \tag{8}$$

Here $def(\boldsymbol{u})$ denotes a tensor of the II rank calculated by the Cauchy formula [12] for small deformation.

We will record the displacement vector as the sum of product their own forms $\boldsymbol{Y}_k$ on of time functions $\varPhi_k(t)$ (*modal approach*):

$$\mathbf{u}(\mathbf{R},t)=\sum_{k=1}^{N}\mathbf{Y}_k(\mathbf{R})\varPhi_k(t)=\Psi(\mathbf{R})\mathbf{\Phi}(t) \\ \Psi(\mathbf{R})^T\Psi(\mathbf{R})=\mathrm{I},\ def(\Psi(\mathbf{R}))^T def(\Psi(\mathbf{R}))=diag(\omega^2) \tag{9}$$

Here I – identity matrix; $def(\Psi(\mathbf{R})$ – vector (column), made up of operators $def(\boldsymbol{Y}_k)$, i.e. each component of this column is the tensor $def(\mathbf{Y}_k)$; $\Psi(\mathbf{R})$ – rectangular matrix which has as many rows as components of the vector **u** and N columns; $diag(\omega^2)$ – diagonal matrix which is composed of squares of natural frequencies.

Substituting in (7) modal decomposition (9) and taking into account properties (8), we obtain integro-differential equations for functions of time $\varPhi_k(t)$, which combine into the vector $\mathbf{\Phi}(t)$:

$$\ddot{\mathbf{\Phi}} + diag\left(\omega^2\right)\mathbf{\Phi} - \int_0^t \mathrm{T}_{\mathrm{mod}}(t-\tau)\mathbf{\Phi}dt = \mathbf{P}(t)$$

$$\mathrm{T}_{\mathrm{mod}}(t) = \int_V def\left[\Psi(\mathbf{R})\right]^T \hat{\hat{\mathrm{T}}}_{dyn}(t) def\left[\Psi(\mathbf{R})\right]dV \tag{10}$$

$$\mathbf{P}(t) = \int_V \rho_0 \Psi^T(\mathbf{R})\mathbf{k}(\mathbf{R},t)dV + \int_S \Psi^T(\mathbf{R})\mathbf{f}(\mathbf{R},t)dS$$

In (10) used the matrix form of record where direct font denotes a rectangular matrix, bold italics – column matrix, the multiplication operations are performed according to the rules of matrix multiplication, the upper symbol T denotes the operation of matrix transposition. When performing matrix operations multiplication the of individual actions above the components of matrices are performed according to the rules prescribed by the meaning component. For example, when calculating the modal matrix relaxation $\mathrm{T}_{\mathrm{mod}}$ its size is determined by the rules of matrix multiplication: a tensor of rank IV $\hat{\hat{\mathrm{T}}}_{dyn}$ is considered as a scalar. Therefore, the dimension of the result is determined by the dimension of the vectors-columns $def\left[\Psi(\mathbf{R})\right]$: i.e. $\mathrm{T}_{\mathrm{mod}}$ there is a square matrix, where the number of rows and columns equal to the number modes, which are retained in the expansion (9) (N×N); each element is calculated according to the rules of tensor multiplication:

$$\left(\mathrm{T}_{\mathrm{mod}}\right)_{k,n} = \int_V def\left(\mathbf{\Psi}_k\right)\cdot\cdot\hat{\hat{\mathrm{T}}}_{dyn}\cdot\cdot def\left(\mathbf{\Psi}_n\right)dV,\ k,n = 1,2,3..N$$

Similarly, when computing the vector of modal forces (the of volume and surface forces **k** and **f**, which are specified in the formula (7) and below) are considered as scalars; here the dimension of the vector of modal forces *P* corresponds to the number of modes specified in (9), i.e. *N*. Components of the modal forces are calculated according to the formula:

$$P_k = \int_V \mathbf{\Psi}_k \cdot \mathbf{k}\rho_0 dV + \int_S \mathbf{\Psi}_k \cdot \mathbf{f}dS$$

The solution of equation (10) is easily obtained by using the integral Laplace transform with respect to time:

$$\mathbf{\Phi}^*(s) = \mathrm{W}^*(s)\left[\mathbf{P}^*(s) - \mathbf{C}(s)\right]$$

$$\mathrm{W}^*(s) = \left[s^2\mathrm{I} + diag\left(\omega^2\right) - \mathrm{T}^*_{\mathrm{mod}}(s)\right]^{-1} \tag{11}$$

$$\mathbf{C}(s) = \int_V \Psi(\mathbf{R})\left[\dot{\mathbf{u}}(\mathbf{R},0) + s\mathbf{u}(\mathbf{R},0)\right]dV = \mathbf{C}_v(0) + s\mathbf{C}_u(0)$$

Here I – identity matrix, s – conversion parameter, $\mathrm{W}^*(s)$ – transfer functions matrix (a term from theory of automatic control), $\mathbf{C}(s)$ – vector, defined as expansion mode Y of coefficients of the initial conditions for displacement $\mathbf{u}(\mathbf{R},0)$ and the speed $\dot{\mathbf{u}}(\mathbf{R},0)$ (when calculating the decomposition of the initial conditions used the same rule as when calculating modal forces, see above).

The solution to (11) is obtained using the theorems of convolution and differentiation of the original [13, 14]:

$$\mathbf{\Phi}(t)=\mathrm{W}(t)\mathbf{C}_v(0)+\dot{\mathrm{W}}(t)\mathbf{C}_u(0)+\int_0^t \mathrm{W}(t-\tau)\mathbf{P}(\tau)d\tau \tag{12}$$

In the last formula the first two terms are the General solution of (10) defined by the initial conditions, the last term (integral Duhamel) is a particular solution corresponding to given loads. Then the function $\mathrm{W}(t)$ is the normalized matrix of fundamental solutions having the obvious properties of (12):

$$\mathrm{W}(0)=0; \quad \dot{\mathrm{W}}(t)=\mathrm{I} \tag{10}$$

Components of tensor of small deformations and the stress tensor can be obtained from expansion mode (9) and physical law (7):

$$\begin{gathered}\hat{\varepsilon}(\mathbf{R},t)=def\left[\mathbf{Y}(\mathbf{R})\right]\mathbf{\Phi}(t)\\ \hat{\sigma}(\mathbf{R},t)=\hat{\hat{\mathrm{E}}}\cdot\cdot def\left[\mathbf{Y}(\mathbf{R})\right]\left[\mathbf{\Phi}(t)-\int_0^t \hat{\hat{\mathrm{T}}}_{dyn}(t-\nu)\mathbf{\Phi}(\tau)d\tau\right]\end{gathered} \tag{11}$$

Thus, if the known oscillation modes Y and normalized matrix of fundamental solutions $\mathrm{W}(t)$, the components of the stress-strain state (SSS) are determined uniquely. If of oscillation modes and the matrix of fundamental solutions is determined analytically, then the components of the SSS are also determined analytically.

4 Conclusion

Dynamic effects of action of short-term loads can be identified by using of modal approach [15, 16] with the assumption of the constancy of the elastic properties of concrete appropriate to the age by the time of application of the disturbing. Superposition of reactions to long - and short-term impacts are possible due to the inadmissibility of finite deformations in building structures.

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