Non-linear shape functions over time in the space-time finite element method

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Abstract. This work presents a generalisation of the space-time finite element method proposed by Kączkowski in his seminal of 1970's and early 1980's works. Kączkowski used linear shape functions in time. The recurrence formula obtained by Kączkowski was conditionally stable. In this paper, non-linear shape functions in time are proposed.

1 Introduction

Kączkowski presented the basis of the space-time finite element method in works [1-4]. The recurrence formula obtained by Kączkowski was conditionally stable. This means that the maximal value of the step of integration in time had to be selected adequately and it had to meet the conditions of method stability, cf.[5]. A very small integration step should be selected in practical dynamical analyses of engineering structures - which is very costly (in terms of the number of numerical operations) and counter-competitive compared to other methods of numerical integration of motion equations, e.g. compared to the Newmark method [6]. Kacprzyk presented a particular method of integration step incrementation [7]. This method requires additional operations to be performed and applies to linear dynamics topics only. Application of a space-time superelement removes the limitation in acceptance of the step of integration in time and allows any integration step to be accepted without loss of the stability of the recurrence formula.

In his work [8], and later, in works [9,10] Kacprzyk presented an unconditionally stable formulation of the space-time finite element method. These works provide a generalisation of the derivation presented by Kączkowski and Langer [4], which is, however, quite different from the basic idea of the method presented in works of Kączkowski [1-3]. In this work, we will use the derivation of Kączkowski presented in work [3]. Kączkowski assumes linear shape functions in time in this fundamental work. An attempt at introduction of a non-linear shape function was presented by Pelc in [11]. The non-linear shape functions assumed in the mentioned work, however, do not lead to a stable recurrence formulae in general, cf. [12].

In this work, we will repeat the derivation presented by Kączkowski in [2] but assuming non-linear shape functions in time for unknown displacements and linear functions for virtual displacements.

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It should be noted that the presented work is not a full generalisation of the space-time finite element method formulated by Kączkowski. This work generalises this method in the so called stationary formulation. The stationary formulation assumes that the division of a structure into finite elements does not change during analysis of the structure. Kączkowski presented in [13] the space-time finite element method, assuming that the division of a structure into elements changes in time. This is the so called non-stationary formulation of the space-time finite element method. This approach was developed by Bajer in [14,15].

The proposed method may be applied, with slight changes, to the non-stationary formulation of the space-time finite element method. A generalisation of this formulation shall provide the basis for our next work.

2 Displacement

Any finite element can be analysed at two time instants: *e* (exordium) and *f* (finis). Combination of appropriate nodes for moments *e* and *f* yields a space-time element (cf. Fig. 1.), cf. Kączkowski [2].

The following notation is adopted: $x_i, i = 1, 2, 3$ - spatial coordinates, t - time coordinate, τ - dimensionless coordinate in time ($\tau = t / h$) with values in the range between -1 and 1 .

Fig. 1. Space-time finite elements.

Fig. 1 presents two- and three-dimensional finite elements. A four-dimensional element cannot be drawn, but all the considerations presented below also apply to four-dimensional elements.

Each finite element in space-time has twice the number of nodes it has in a one-, two- or three-dimensional space.

Displacements of nodes in a finite space-time element form a vector, which can be decomposed as below:

$$\mathbf{q}_e = \left\{ \begin{matrix} \mathbf{q}_e^e \\ \mathbf{q}_e^f \end{matrix} \right\} \tag{1}$$

Displacements in space-time can be expressed conventionally:

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{N}(\mathbf{x}, t) \mathbf{q}_e \tag{2}$$

where

$$\mathbf{x} = \left\{ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\} \tag{3}$$

The shape matrix $\mathbf{N}$ will be written as a tensor product of two matrices

$$\mathbf{N}(\mathbf{x}, t) = \mathbf{N}_t(t) \otimes \mathbf{N}_x(\mathbf{x}) \quad (4)$$

In this product, the $\mathbf{N}_t(t)$ matrix has a dimensions 1×2 and depends on the time coordinate t only, whilst the $\mathbf{N}_x(\mathbf{x})$, matrix, dependent on spatial coordinates, is a typical shape matrix of an element applied to static problems.

3 Strains and stresses

The vector of strain components $\boldsymbol{\varepsilon}_x$ is obtained by appropriate differentiations over spatial coordinates of displacements vector $\mathbf{f}$.

$$\boldsymbol{\varepsilon}_x = \partial_x \mathbf{f} = \partial_x (\mathbf{N}_t \otimes \mathbf{N}_x) \mathbf{q}_e = (\mathbf{N}_t \otimes \partial_x \mathbf{N}_x) \mathbf{q}_e \quad (5)$$

denoting

$$\mathbf{B}_x = \partial_x \mathbf{N}_x \quad (6)$$

we eventually obtain

$$\boldsymbol{\varepsilon}_x = (\mathbf{N}_t \otimes \mathbf{B}_x) \mathbf{q}_e \quad (7)$$

A finite space-time element is also subjected to "space-time strains" $\boldsymbol{\varepsilon}_t$, namely to motion velocities. These are obtained by differentiating the displacements in time.

$$\boldsymbol{\varepsilon}_t = \frac{\partial}{\partial t} \mathbf{f} = \frac{\partial}{\partial t} (\mathbf{N}_t \otimes \mathbf{N}_x) \mathbf{q}_e = \left(\frac{d}{dt} \mathbf{N}_t \otimes \mathbf{N}_x \right) \mathbf{q}_e \quad (8)$$

By assuming the following notation

$$\dot{\mathbf{N}}_t = \frac{d}{dt} \mathbf{N}_t \quad (9)$$

we obtain

$$\boldsymbol{\varepsilon}_t = (\dot{\mathbf{N}}_t \otimes \mathbf{N}_x) \mathbf{q}_e \quad (10)$$

Generalised internal forces comprising a single column matrix $\boldsymbol{\sigma}_x$ depend on element strains. Assuming that the mechanical properties of a viscous-elastic material can be modeled using the Kelvin-Voigt approach, we can write

$$\boldsymbol{\sigma}_x = (\mathbf{E} + \boldsymbol{\eta}_w \frac{\partial}{\partial t}) \boldsymbol{\varepsilon}_x, \quad (11)$$

where $\mathbf{E}$ - elastic constants matrix, $\boldsymbol{\eta}_w$ - square matrix containing viscosity coefficients.

Substituting relationships (7) in to the formula (11) we obtain

$$\boldsymbol{\sigma}_x = [\mathbf{N}_t \otimes (\mathbf{E} \mathbf{B}_x)] \mathbf{q}_e + [\dot{\mathbf{N}}_t \otimes (\boldsymbol{\eta}_w \mathbf{B}_x)] \mathbf{q}_e. \quad (12)$$

In addition to these conventional, internal forces, some additional quantities are present in space-time, namely momentums (cf. [1,2]). These are obtained from the relationship:

$$\boldsymbol{\sigma}_t = -\mathbf{m} \boldsymbol{\varepsilon}_t = -[\dot{\mathbf{N}}_t \otimes (\mathbf{m} \mathbf{N}_x)] \mathbf{q}_e, \quad (13)$$

where $\mathbf{m}$ is a square matrix of masses.

4 Space-time stiffness matrix

According to the virtual four-work theorem, cf. [1], and taking into account linear, viscous external damping, we can obtain the following equation:

$$\bar{\mathbf{q}}_e^T \mathbf{Q}_e + \int_{-1}^1 \int_V \bar{\mathbf{f}}^T (\mathbf{p} - \boldsymbol{\eta}_e \frac{\partial}{\partial t} \mathbf{f}) dV h d\tau = \int_{-1}^1 \int_V (\boldsymbol{\varepsilon}_x^T \boldsymbol{\sigma}_x + \boldsymbol{\varepsilon}_t^T \boldsymbol{\sigma}_t) dV h d\tau, \quad (14)$$

where V is a volume of the element in a three-dimensional space.

The virtual state of a translation may be defined as follows, analogously to (2):

$$\bar{\mathbf{f}}(\mathbf{x}, t) = [\mathbf{W}_t(t) \otimes \mathbf{N}_x(x)] \bar{\mathbf{q}}_e \quad (15)$$

The shape matrix $\mathbf{W}_t(t)$ is a function of the time variable. Thus, and per analogy to (7) and (10) we obtain

$$\bar{\boldsymbol{\varepsilon}}_x = (\mathbf{W}_t \otimes \mathbf{B}_x) \bar{\mathbf{q}}_e, \quad (16)$$

$$\bar{\boldsymbol{\varepsilon}}_t = (\dot{\mathbf{W}}_t \otimes \mathbf{N}_x) \bar{\mathbf{q}}_e, \quad (17)$$

where

$$\dot{\mathbf{W}}_t = \frac{d}{dt} \mathbf{W}_t \quad (18)$$

By executing the assigned substitution and using the formula (14) we obtain

$$\mathbf{Q}_e = \mathbf{K}_e \mathbf{q}_e - \mathbf{Q}_{ep} \quad (19)$$

where

$$\begin{aligned} \mathbf{K}_e = & \int_{-1}^1 \mathbf{W}_t^T \mathbf{N}_t h d\tau \otimes \int_V \mathbf{B}_x^T \mathbf{E} \mathbf{B}_x dV - \int_{-1}^1 \dot{\mathbf{W}}_t^T \dot{\mathbf{N}}_t h d\tau \otimes \int_V \mathbf{N}_x^T \mathbf{m} \mathbf{N}_x dV + \\ & + \int_{-1}^1 \mathbf{W}_t^T \dot{\mathbf{N}}_t h d\tau \otimes \left(\int_V \mathbf{B}_x^T \boldsymbol{\eta}_w \mathbf{B}_x dV + \int_V \mathbf{N}_x^T \boldsymbol{\eta}_z \mathbf{N}_x dV \right) \end{aligned} \quad (20)$$

and

$$\mathbf{Q}_{ep} = \int_{-1}^1 \int_V (\mathbf{W}_t^T \otimes \mathbf{N}_x^T) p dV h d\tau \quad (21)$$

Let us introduce the following notation:

$$\begin{aligned} \mathbf{S}_e &= \int_V \mathbf{B}_x^T \mathbf{E} \mathbf{B}_x dV, \quad \mathbf{M}_e = \int_V \mathbf{N}_x^T \mathbf{m} \mathbf{N}_x dV \\ \mathbf{C}_e^w &= \int_V \mathbf{B}_x^T \boldsymbol{\eta}_w \mathbf{B}_x dV, \quad \mathbf{C}_e^z = \int_V \mathbf{N}_x^T \boldsymbol{\eta}_z \mathbf{N}_x dV \end{aligned} \quad (22)$$

The $\mathbf{S}_e$ is a static stiffness matrix of the element $\mathbf{M}_e$ - a matrix of element mass, $\mathbf{C}_e^w$ and $\mathbf{C}_e^z$ are matrices of internal and external damping, respectively.

Taking into account the relationships (22) formula (20) takes the following form:

$$\mathbf{K}_e = \int_{-1}^1 \mathbf{W}_t^T \mathbf{N}_t h d\tau \otimes \mathbf{S}_e - \int_{-1}^1 \dot{\mathbf{W}}_t^T \dot{\mathbf{N}}_t h d\tau \otimes \mathbf{M}_e + \int_{-1}^1 \mathbf{W}_t^T \dot{\mathbf{N}}_t h d\tau \otimes (\mathbf{C}_e^w + \mathbf{C}_e^z) \quad (23)$$

We assume non-linear shape functions in time $\mathbf{N}_t$, cf. Fig. 2.

$$\mathbf{N}_t = \left[\frac{1}{2}(1-\tau) + \varsigma(\tau-\tau^3), \frac{1}{2}(1+\tau) - \varsigma(\tau-\tau^3) \right] \quad (24)$$

where ς is a number (parameter), whilst $\tau = -1$ at time e and $\tau = 1$ at time f.

On the other hand, in the case of virtual displacements $\mathbf{W}_t$, we will assume linear shape functions, cf. Fig. 3.

$$\mathbf{W}_t = \frac{1}{2} [1-\tau, 1+\tau] \quad (25)$$

Derivatives of the shape function in time are as follows:

$$\dot{\mathbf{N}}_t = \frac{1}{h} \left[-\frac{1}{2} + \varsigma(\tau-3\tau^2), \frac{1}{2} - \varsigma(\tau-3\tau^2) \right]. \quad (26)$$

and

$$\dot{\mathbf{W}}_t = \frac{1}{2h}[-1, 1] \tag{27}$$

Fig.2. Shape functions in time for various ζ values

Fig. 3. Shape functions in time for virtual displacements

Integrals present in formula (23) can be easily calculated. Thus

$$\int_{-1}^1 \mathbf{W}_t^T \mathbf{N}_t h d\tau = 2h \begin{bmatrix} \frac{1}{3} - \frac{1}{15}\varsigma & \frac{1}{6} + \frac{1}{15}\varsigma \\ \frac{1}{6} + \frac{1}{15}\varsigma & \frac{1}{3} - \frac{1}{15}\varsigma \end{bmatrix}, \quad (28)$$

$$\int_{-1}^1 \dot{\mathbf{W}}_t^T \dot{\mathbf{N}}_t h d\tau = \frac{1}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \int_{-1}^1 \mathbf{W}_t^T \dot{\mathbf{N}}_t h d\tau = \frac{1}{2h} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

It should be noted that the reverse substitution, that of a non-linear shape function for virtual displacements and of a linear shape function for real displacements leads to the same integration results.

By using the substitution (such substitutions facilitate comparisons of the resulting formulae with other methods)

$$\varsigma = 15\left(\beta - \frac{1}{6}\right) \quad (29)$$

we obtain

$$\mathbf{K}_e = 2h \begin{bmatrix} \frac{1}{2} - \beta & \beta \\ \beta & \frac{1}{2} - \beta \end{bmatrix} \otimes \mathbf{S}_e - \frac{1}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \mathbf{M}_e + \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \otimes (\mathbf{C}_e^w + \mathbf{C}_e^z). \quad (30)$$

Let

$$\mathbf{K}_e = \begin{bmatrix} \mathbf{A}_e & \mathbf{B}_e \\ \mathbf{B}_e^i & \mathbf{D}_e \end{bmatrix}. \quad (31)$$

Thus

$$\begin{aligned} \mathbf{A}_e &= 2h\left(\frac{1}{2} - \beta\right)\mathbf{S}_e - \frac{1}{2h}\mathbf{M}_e - \frac{1}{2}(\mathbf{C}_e^w + \mathbf{C}_e^z), & \mathbf{B}_e &= 2h\beta\mathbf{S}_e + \frac{1}{2h}\mathbf{M}_e + \frac{1}{2}(\mathbf{C}_e^w + \mathbf{C}_e^z), \\ \mathbf{B}_e^i &= 2h\beta\mathbf{S}_e + \frac{1}{2h}\mathbf{M}_e - \frac{1}{2}(\mathbf{C}_e^w + \mathbf{C}_e^z), & \mathbf{D}_e &= 2h\left(\frac{1}{2} - \beta\right)\mathbf{S}_e - \frac{1}{2h}\mathbf{M}_e + \frac{1}{2}(\mathbf{C}_e^w + \mathbf{C}_e^z) \end{aligned} \quad (32)$$

Let us also introduce the following notation:

$$\mathbf{A}_e + \mathbf{D}_e = \mathbf{H}_e = 4h\left(\frac{1}{2} - \beta\right)\mathbf{S}_e - \frac{1}{h}\mathbf{M}_e \quad (33)$$

5 System solution and recurrent formulae

Space-time elements contained between instants e and f can be combined into a structure observed within the same time range. We thus obtain a space-time element of a single time layer. Matrices $\mathbf{S}$, $\mathbf{M}$, $\mathbf{C}^w$, $\mathbf{C}^z$ for the entire structure must be formed in order to obtain the stiffness matrix for such a space-time element. Space-time elements for individual time layers are then combined and the stiffness matrix for the space-time structure over the entire observation time is observed.

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & & \\ \mathbf{B}^i & \mathbf{H} & \mathbf{B} & \\ & \mathbf{B}^i & \mathbf{H} & \mathbf{B} \\ & & \dots & \dots & \dots \end{bmatrix} \quad (34)$$

The number of rows in matrix $\mathbf{K}$ is equal to the number of time layers. Finally, the system of equations of the space-time element method has the following form.

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & & \\ \mathbf{B}^i & \mathbf{H} & \mathbf{B} & \\ & \mathbf{B}^i & \mathbf{H} & \mathbf{B} \\ & & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \mathbf{q}^0 \\ \mathbf{q}^1 \\ \mathbf{q}^2 \\ \dots \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^0 \\ \mathbf{Q}^1 \\ \mathbf{Q}^2 \\ \dots \end{bmatrix}, \quad (35)$$

where $\mathbf{q}^i$, $\mathbf{Q}^i$ denote vectors of displacements and of impulses, respectively. The dimension of these vectors is equal to the number of degrees of freedom of the structure.

While

$$\begin{aligned} \mathbf{A} &= h(1-2\beta)\mathbf{S} - \frac{1}{2h}\mathbf{M} - \frac{1}{2}(\mathbf{C}^w + \mathbf{C}^z), \\ \mathbf{B}, \mathbf{B}^i &= 2h\beta\mathbf{S} + \frac{1}{2h}\mathbf{M} \pm \frac{1}{2}(\mathbf{C}^w + \mathbf{C}^z), \\ \mathbf{H} &= 2h(1-2\beta)\mathbf{S} - \frac{1}{h}\mathbf{M}. \end{aligned} \quad (36)$$

System (35) is solved using the recurrent formula:

$$\begin{aligned} \mathbf{B}\mathbf{q}^1 &= \mathbf{Q}^0 - \mathbf{A}\mathbf{q}^0, \\ \mathbf{B}\mathbf{q}^{i+1} &= \mathbf{Q}^i - \mathbf{H}\mathbf{q}^i - \mathbf{B}^i\mathbf{q}^{i-1}, \quad i = 1, 2, 3, \dots \end{aligned} \quad (37)$$

It should be noted that for $\beta = 0$ the stiffness matrix $\mathbf{K}$ is multiplied by zero (cf. relationship (36)) and does not affect the values of components the $\mathbf{B}, \mathbf{B}^i$ matrices. In the case of undamped vibrations and assuming the diagonality of the mass matrix $\mathbf{M}$ an explicit recurrent diagram is obtained - the matrix of $\mathbf{B}$ indices is diagonal.

Displacements are the unknown variables in the recurrent scheme described according to equation (37). Velocities and accelerations are obtained from the respective differential equations, cf. [8,10].

Assuming $\varsigma = 0$ we obtain $\mathbf{W}_t = \mathbf{N}_t$ and $\beta = 1/6$. Formulae (36) and (37) are then identical to the formulae obtained in the work of Kączkowski [2].

6 Summary

The recurrent formulae obtained in this work have been studied by numerous authors, [16]. Numerical verifications show that no single best scheme can be indicated. The unconditionally stable scheme is the most popular one. However, errors in the solution resulting from assumption of an excessive integration step should be taken into account when using this algorithm, cf. [9,10].

Work [10] included numerical testing of the aforementioned formulae. Each of the proposed recurrent formulae leads to correct results. The recurrent formulae differ mostly in the permitted integration step in time (a $2h$ time step) and in costs of the solution (understood as the number of numerical operations). There are also some slight differences in solutions caused by numerical dumping. However, these differences are insignificant in technical applications.

Work [5] presents a wide description of various numerical integration methods applied to the motion equation. This work provided an analytical proof of equivalence of numerous recurrent formulae. According to this work, it may be shown that for $\beta = 1/4$ the generalised space-time finite element method is identical to the Newmark method. On the other hand, for $\beta = 0$ and assuming the mass matrix $\mathbf{M}$ and taking damping into account,

as well as diagonal damping matrices $\mathbf{C}^w$ and $\mathbf{C}^z$ an explicit recurrent diagram is obtained. This means that we no longer have to solve a system of equations during the recurrence.

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