

# Secondary consolidation modelling by using rheological schemes

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**Abstract.** The paper presents the possibility of the secondary consolidation modelling in soil mechanics by using rheological schemes. Non-elastic rheological models represent time-dependent behaviour of the soil skeleton and should be expressed by appropriate constitutive relationships in the form of ordinary differential equations. On the other hand, the primary consolidation phenomenon caused by dissipation of the excess pore pressure is modelled by applying the well-known formulation proposed by Terzaghi and expressed by the partial differential equation. The resulting initial boundary value problem was solved numerically for three selected rheological schemes. Algorithms implemented in Mathematica were used for this purpose.

## 1 Introduction

In order to provide some physical insight into non-elastic and time-dependent behaviour of engineering materials, rheological schemes consisting of springs, dashpots and sliders are used very often [1]. In case of many viscoelastic materials an accurate scheme is the Standard Model, which consists of a spring in series with a parallel combination of a spring and dashpot. Viscoelastic properties of asphaltic materials can be modelled by using the Burgers scheme (Maxwell and Kelvin-Voigt in series) [2]. The application of rheological schemes consisting of additional sliders, gives the possibility of modelling plasticity phenomenon making a material capable to exhibit additional permanent deformations.

Rheological schemes can also be applied to model a consolidation problem in soil mechanics. This time-dependent process depends on permeability and compressibility of the soil as well as the length of the drainage path. Generally, consolidation process consists of two parts: (i) primary consolidation settlement, caused by dissipation of the excess pore pressure generated by load application and (ii) secondary compression, caused by the time-dependent deformation behaviour of soil particles [3].

The theory of one-dimensional consolidation was first proposed by Terzaghi (1925) basing on the linear elastic relationship between effective stress and strain [4]. Such an approach describes only the primary consolidation process. By using non-elastic and time-dependent rheological schemes, modelling the behaviour of the soil skeleton, it is possible to describe secondary consolidation [5-8].

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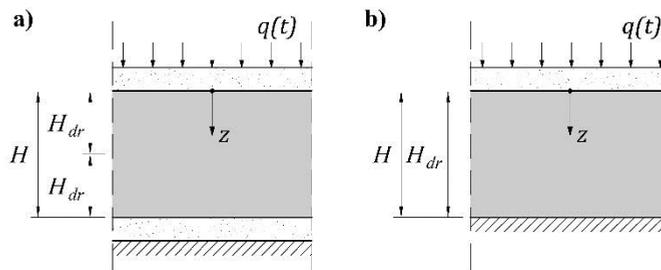
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The main goal of this paper is to demonstrate various rheological schemes suited for the secondary consolidation modelling. The systems of differential equations, defining the initial boundary value problem (IBVP) in case of one-dimensional consolidation is formulated. Numerical solution of the IBVP was obtained by using algorithms implemented in Mathematica [9].

## 2 One-dimensional consolidation problem

In soil mechanics, one-dimensional consolidation problem is considered as a consolidation of an infinite saturated homogeneous clay layer, loaded by an infinite and uniform pressure. The soil layer exhibiting consolidation process can be drained on both sides (Fig. 1a), when it is located between two layers of granular soil, or on one side (Fig. 1b). In either cases volume change of the granular soil layers is neglected and the whole half-space rests on a non-deformable rock formation. Aforementioned simplification leads to, so called, oedometer conditions in which horizontal strains are zero and the change in volume of the clay layer is related only to its vertical strain. For a two-side drainage, the flow distance  $H_{dr} = H/2$ .

For simplification, we consider only excess pore water pressure which arises after application of the external load (the uniform pressure), neglecting the overburden pressure state.



**Fig. 1.** One-dimensional consolidation problem: a) two-side drainage, b) one-side drainage.

Continuity of flow (filtration) in the cohesive soil layer along with the Darcy's law gives the well-known differential equation of one-dimensional consolidation

$$-\frac{k}{\gamma_w} \frac{\partial^2 u(z,t)}{\partial z^2} = \frac{1}{1+e_0} \frac{\partial e(z,t)}{\partial t} \quad (1)$$

where:  $k$  – coefficient of permeability [m/s],  $\gamma_w$  – unit weight of water [N/m<sup>3</sup>],  $e_0$  – initial void ratio,  $u(z,t)$  and  $e(z,t)$  – functions of pore water pressure and void ratio respectively.

Applying a relationship between the vertical strain and the void ratio  $e = \varepsilon(1+e_0) + e_0$ , Eq. (1) can be rewritten as follows

$$\frac{k}{\gamma_w} \frac{\partial^2 u(z,t)}{\partial z^2} = -\frac{\partial \varepsilon(z,t)}{\partial t} \quad (2)$$

In order to consider different loading programmes, Eq. (2) has to be formulated in effective stress  $\sigma'$ . Assuming that for 1D-consolidation, the applied total stress  $\sigma(z,t)$  can vary only linearly with depth, substituting pore pressure  $u = \sigma - \sigma'$  gives

$$\frac{k}{\gamma_w} \frac{\partial^2 \sigma'(z,t)}{\partial z^2} = \frac{\partial \varepsilon(z,t)}{\partial t} \tag{3}$$

Setting a linear constitutive relationship for the soil skeleton, for example  $\varepsilon = \sigma' / M_0$ , where  $M_0$  is soil's oedometric modulus, we can model primary consolidation phenomenon which is related only to dissipation of pore water pressure and drainage. The modulus  $M_0$  can be determined from a simple oedometer consolidation laboratory test. For a one-side drainage, Eq. (3) has to be solved with the following boundary conditions (see Fig. 1b)

$$\left. \frac{\partial \sigma'}{\partial z} \right|_{z=H} = 0 \quad \sigma' |_{z=0} = q(t) \quad 0 < t \leq \infty \tag{4}$$

where  $q(t)$  is a function of the applied external pressure.

Classically, the above problem was solved for a constant pressure, without unloading. The solution for excess pore water pressure, assuming the boundary condition  $\sigma' |_{z=0} = q_0 = const.$  is shown in Fig. 2.

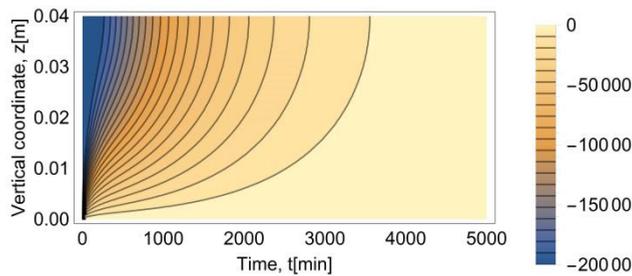


Fig. 2. Contours of excess pore water pressure for primary consolidation (linear constitutive relationship for soil skeleton).

### 3 Secondary consolidation modelling

In order to take into account secondary consolidation effects, related to time-dependent behaviour of the soil skeleton, the constitutive relationship in the space of the effective stress can be formulated by using rheological schemes (see Fig. 3). Non-elastic constitutive relationships of the soil skeleton should be expressed in the form of ordinary differential equations with respect to time.

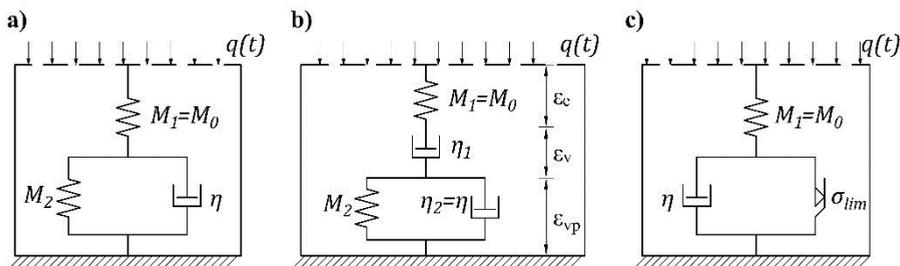


Fig. 3. Rheological schemes applied for soil skeleton: a) Standard Model, b) Burgers Model, c) Bingham Model.

Gibson and Lo [6] presented an analytical solution of one-dimensional consolidation in which the relationship between the effective stress and vertical strain was formulated according to the Standard Model shown in Fig. 3a. Viscoelastic constitutive relationship of this model can be expressed as follows

$$\eta M_0 \frac{\partial \varepsilon}{\partial t} + M_0 M_2 \varepsilon = \eta \frac{\partial \sigma'}{\partial t} + (M_0 + M_2) \sigma' \quad (5)$$

The upper spring shown in Fig. 3a accounts for the primary consolidation and the bottom viscoelastic part represents the secondary consolidation. In order to find the function of vertical strain  $\varepsilon(z, t)$ , the system of differential equations (3) and (5) has to be solved with the set of boundary conditions (4) and the following initial condition

$$\varepsilon(z, t)|_{t=0} = 0 \quad (6)$$

Solving the IBVP allows the function of displacement (settlement)  $s(z, t)$  to be found by integrating the strain function

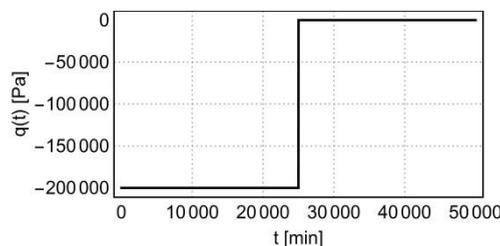
$$\frac{\partial}{\partial z} s(z, t) = \varepsilon(z, t) \quad s(z, t)|_{z=H} = 0 \quad (7)$$

**Tab. 1.** Parameters assumed for the comparison of different models.

$k$	$M_1 = M_0$	$M_2$	$\eta = \eta_2$	$\eta_1$	$H$	$q_0$	$\sigma_{lim}$
[cm/s]	[kPa]	[kPa]	[Ns/m <sup>2</sup> ]	[Ns/m <sup>2</sup> ]	[cm]	[kPa]	[kPa]
$10^{-8}$	1000	5000	$5 \cdot 10^{10}$	$1 \cdot 10^{11}$	40.0	200.0	50.0

The analytical solution presented in [6] was used to test a numerical solution. The system of partial differential equations was solved using the numerical method of lines implemented in Mathematica software [9]. The resulting system of ordinary differential equations was solved using LSODA method in Mathematica [9]. For the numerical solution, the following loading programme was assumed (Fig. 4):

$$q(t) = \begin{cases} q_0 & 0 \leq t \leq t_0 \\ 0 & t < 0 \wedge t > t_0 \end{cases} \quad (8)$$



**Fig. 4.** Loading programme.

Fig. 5 presents comparison between numerical and analytical solutions. Assumed material parameters are shown in Tab. 1. It should be emphasised, however, that these parameters were chosen to present only the capabilities of the approach. In the future work curve-fitting of more complicated models to actual laboratory test data is planned.

In case of the Burgers model (Fig. 3b), the constitutive relationship can be formulated as follows

$$\eta M_0 \frac{\partial}{\partial t} \varepsilon + M_0 M_2 \varepsilon = \eta \frac{\partial}{\partial t} \sigma' + M_2 \sigma' + M_0 M_2 \varepsilon_v \tag{9}$$

where the additional internal variable  $\varepsilon_v$  should be evaluated by the following differential equation

$$\frac{\partial}{\partial t} \varepsilon_v = \frac{1}{\eta_1} \sigma', \quad \varepsilon_v(z, t)|_{t=0} = 0 \tag{10}$$

The system of differential equations (3), (9) and (10) was solved with the boundary conditions (4) and zero initial conditions for  $\varepsilon$  and  $\varepsilon_v$  (see Fig. 5).

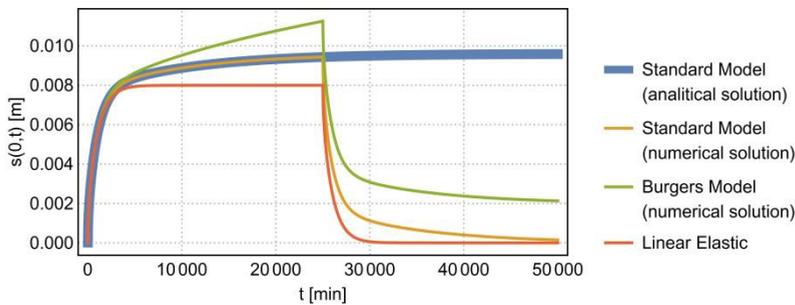


Fig. 5. Comparison of settlement curves for different solutions and viscoelastic rheological models.

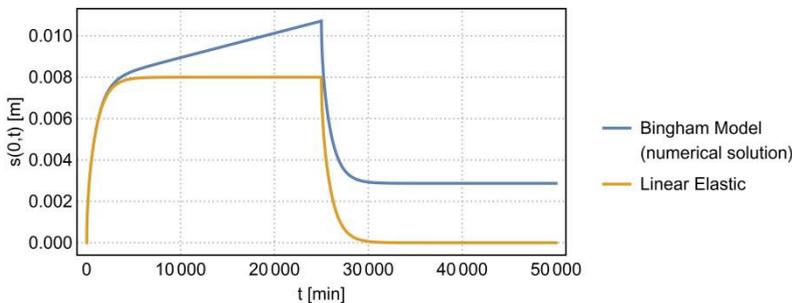


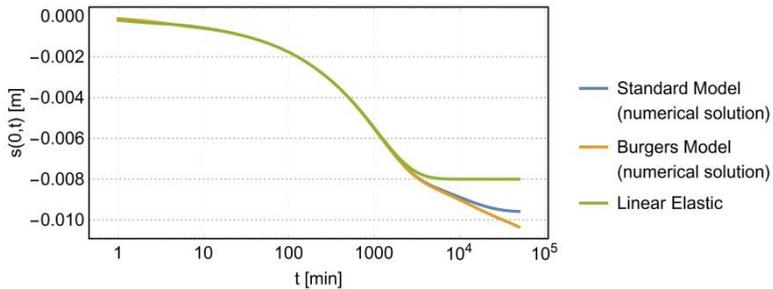
Fig. 6. Comparison of settlement curves between linear elastic solution and Bingham Model.

Let us analyse another rheological scheme, called the Bingham model, representing elastic-visco-plastic behaviour of the soil skeleton (Fig. 3c). Constitutive relationship defining this model can be expressed by the following non-linear equation

$$\frac{\partial}{\partial t} \varepsilon = \begin{cases} \frac{1}{M_0} \frac{\partial}{\partial t} \sigma' & \text{if } |\sigma'| < \sigma_{lim} \\ \frac{1}{M_0} \frac{\partial}{\partial t} \sigma' + \frac{|\sigma'| - \sigma_{lim}}{\eta} \text{sgn}(\sigma') & \text{if } |\sigma'| \geq \sigma_{lim} \end{cases} \tag{11}$$

The solution of the IBVP for the Bingham Model is visualized in Fig. 6, showing a difference between non-elastic behaviour of the soil skeleton and elastic one.

Finally, the comparison of viscoelastic rheological schemes analysed in our paper is shown in Fig. 7. For this calculation,  $\eta_1 = 5 \cdot 10^{11} \text{Ns/m}^2$  and  $q(t) = q_0 = \text{const}$  were assumed in order to present the solution in a way similar to oedometric laboratory test results.



**Fig. 7.** Comparison of settlement curves for viscoelastic rheological models shown on semi-logarithmic graph.

## 4 Conclusions

The mathematical formulation of the soil mechanics consolidation problem was presented in the paper. Viscoelastic and elastic-visco-plastic rheological schemes can be used in order to represent secondary consolidation of the soil skeleton. The results of numerical solutions demonstrate the validity of the proposed method.

The realistic modelling of engineering materials needs a three-dimensional description to be applied which leads to the strain-stress relationships involving the notion of tensors [10, 11]. The authors are currently underway the work in the area of 3D constitutive modelling of consolidation problems combining Finite Element Method simulations along with laboratory experiments.

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