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Analytical model of unsteady-state convective heat transfer between the heat carrier and the finite sizes plate adjusted for the thermal relaxation

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Abstract. A hyperbolic boundary value problem of the thermal conduction of a two-dimensional plate with the third kind boundary conditions is formulated. The transient thermal process in the plate is due to the temperature changes of the external medium over time and along the plate length, and also by a multiple step change of the plate surface heat transfer coefficient throughout the transient process. An analytical solution with improved convergence adjusted for thermal relaxation and thermal damping is obtained for the temperature field in the plate.

1 Introduction

In short-term transient processes, the results of calculations using the classical differential thermal conductivity equation

$$\partial T/\partial\tau = a\,\Delta T \tag{1}$$

are characterized by large deviations from real processes due to neglect of thermal inertia. This specificity of equation (1) was noted by such scientists as Maxwell [1], Onzager [2], Cattaneo [3], Vernotte [4], Luikov [5] et al., and was confirmed experimentally (Fig. 1).

To overcome the limitations of Fourier formula

$$\mathbf{q} = -\lambda\,\mathbf{grad}\,T\,,$$

Luikov [5], Tzou [6] et al proposed to take into account the effect of relaxation phenomena to heat conduction, due to which Fourier formula takes the form of expression, also known as the Maxwell-Cattaneo-Luikov equation [5, 7, 8] or the dual-phase-lag Eq. [6, 9, 10]:

$$\mathbf{q} + \tau_q\,\partial\mathbf{q}/\partial\tau = -\lambda\,\mathbf{grad}\left(T + \tau_T\,\partial T/\partial\tau\right), \tag{2}$$

owing to what equation (1) was transformed into a differential equation of heat conduction of hyperbolic type

$$\partial T/\partial\tau + \tau_q\,\partial^2 T/\partial\tau^2 = a\,\Delta\left(T + \tau_T\,\partial T/\partial\tau\right). \tag{3}$$

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Here τ_q is the thermal relaxation time, which characterizes the property of thermal inertia or thermal "elasticity" of the body, s; τ_T is the temperature damping time, which characterizes the reaction of temperature on changing of heat flux, s.

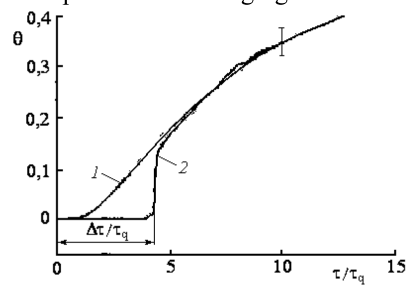

Fig. 1. Comparison of the experimental and calculated transient thermal processes using Fourier hypothesis: *1*-solution data (1); *2* - experimental data [9].

In [7-11] the analytical calculations of transient thermal processes taking into account equation (3) with sufficiently significant simplifications are presented. In [7, 8] models of transient thermal processes in a solid were constructed for a sudden change of the external medium temperature, which was assumed constant with an unchanged heat transfer coefficient during the entire transient process [10], or during individual time intervals (periods) [11].

In this paper a plate of thickness $2h$ and length l is considered. It is assumed, that the heat transfer coefficient of the plate varies in time stepwise, and the medium temperature (heat carrier) in a separate period depends on the time and length of the plate in the polynomial form:

$$\theta_{f,j}(Y,t)=\sum_{k=0}^{k_y} Y^k \sum_{l=0}^{k_t} g_{k,l} t^l ,$$

where $g_{k,l}$ is the coefficient; $t=\tau/\tau_q$ is the relative time; $Y=y/l$ is the relative longitudinal coordinate; k_y and k_t are the counts of members of each row.

2 The third kind hyperbolic boundary value problem

The mathematical formulation of the boundary value problem of the thermal conductivity of a plate with boundary conditions in the form (2) and the differential equation (3) in dimensionless variables:

$$\frac{\partial^2\theta_j}{\partial t^2}+\frac{\partial\theta_j}{\partial t}=\mathrm{Fo}_{q,j}\left[\frac{\partial^2}{\partial X^2}\left(\theta_j+\kappa\frac{\partial\theta_j}{\partial t}\right)+\frac{1}{L^2}\frac{\partial^2}{\partial Y^2}\left(\theta_j+\kappa\frac{\partial\theta_j}{\partial t}\right)\right]; \qquad (4)$$

$$\theta_j(X,Y,0)=\begin{cases}1 & \text{for } j=1,\\ \theta_{j-1}(X,Y,t_{p,j-1}) & \text{for } j>1;\end{cases} \quad \frac{\partial\theta_j(X,Y,0)}{\partial t}=\begin{cases}0 & \text{for } j=1,\\ \partial\theta_{j-1}(X,Y,t_{p,j-1})/\partial t & \text{for } j>1;\end{cases} \qquad (5)$$

$$\partial\theta_j(0,Y,t)/\partial X=0\,; \qquad (6)$$

$$\partial\left[\theta_j(1,Y,t)+\kappa\,\partial\theta_j(1,Y,t)/\partial t\right]/\partial X=-\mathrm{Bi}_{x,j}\left[\theta_j(1,Y,t)-\theta_{f,j}(Y,t)+\partial\theta_j(1,Y,t)/\partial t\right]; \qquad (7)$$

$$\partial\left[\theta_j(X,0,t)+\kappa\partial\theta_j(X,0,t)/\partial t\right]/\partial X=\mathrm{Bi}_{y=0,j}\left[\theta_j(X,0,t)-\theta_{f,j}(0,t)+\partial\theta_j(X,0,t)/\partial t\right]; \qquad (8)$$

$$\partial\left[\theta_j(X,1,t)+\kappa\partial\theta_j(X,1,t)/\partial t\right]/\partial Y=-\mathrm{Bi}_{y=1,j}\left[\theta_j(X,1,t)-\theta_{f,j}(1,t)+\partial\theta_j(X,1,t)/\partial t\right]. \qquad (9)$$

Here $\theta=(T-T^*)/(T_0-T^*)$; T^* is the characteristic temperature, K; T_0 is the initial plate temperature, K; j = 1, 2, … is the period number; $X=x/h$; x is the transverse coordinate relative to the median plane, m; $L = l/h$; $\kappa = \tau_T/\tau_q$; $\mathrm{Fo}_{q,j} = a_j\tau_q/h^2$; $a_j = \lambda_j/(\rho c_j)$; $\mathrm{Bi}_{x,j} = \alpha_j h/\lambda_j$, $\mathrm{Bi}_{y=0,j} = \alpha_{y=0,j} l/\lambda_j$ and $\mathrm{Bi}_{y=1,j} = \alpha_{y=1,j} l/\lambda_j$ are the Bio numbers; α_j, $\alpha_{y=0,j}$ and $\alpha_{y=1,j}$ is the heat transfer coefficient in the *j*-th period, W/(m^2 K) and $t_{p,j} = \tau_{p,j}/\tau_q$ is the relative duration of the *j*-th period.

2.1. An analytic solution of the boundary value problem

The solution is obtained using the finite Fourier integral transforms [11, 12]:

$$\theta_j(X,Y,t) = \sum_{n=1}^{\infty} A_{n,j} K_x(\mu_{n,j}X) \sum_{m=1}^{\infty} A_{m,j} K_y(\gamma_{m,j}Y)\theta_{L,j}(\mu_{n,j},\gamma_{m,j},t), \tag{10}$$

where $A_{n,j}^{-1} \equiv \int_0^1 K_x^2(\mu_{n,j}X)\,dX$; $A_{m,j}^{-1} \equiv \int_0^1 K_y^2(\gamma_{m,j}Y)\,dY$;, $\vartheta_j^2 = (\mu_{n,j}^2 + \gamma_{m,j}^2/L^2)$; $\beta_j = 1 + \kappa\,\mathrm{Fo}_{q,j}\vartheta_j^2$

$\zeta_j^2 = \left|\beta_j^2 - 4\,\mathrm{Fo}_{q,j}\,\vartheta^2\right|$; $\theta_{L,j}(\mu_{n,j},\gamma_{m,j},t) = C_{1,j}\,f_{1,j}(t) + C_{2,j}\,f_{2,j}(t) + \mathrm{Fo}_{q,j}F_j(t)$; $W_j(t) = \sum_{l=0}^{k_t} d_l t^l + \sum_{l=1}^{k_t} l\,d_l t^{l-1}$

$$f_{1,j}(t) = \begin{cases} t\exp(-\beta_j t/2) & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 = \beta_j^2/4, \\ \exp(-a_1 t) & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 < \beta_j^2/4, \\ \exp(-\beta_j t/2)\cos(\zeta_j\,t/2) & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 > \beta_j^2/4; \end{cases}$$

$$f_{2,j}(t) = \begin{cases} \exp(-\beta_j\,t/2) & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 = \beta_j^2/4, \\ \exp(-a_2 t) & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 < \beta_j^2/4, \\ \exp(-\beta_j\,t/2)\sin(\zeta_j\,t/2) & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 > \beta_j^2/4; \end{cases}$$

$$F_j(t) = \begin{cases} \int_0^t W_j(\eta)(t-\eta)\exp\left[-\beta_j(\eta - t)/2\right]d\eta & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 = \beta_j^2/4, \\ \dfrac{2}{\zeta_j}\int_0^t W_j(\eta)\,\mathrm{sh}\left[\zeta_j(t-\eta)/2\right]\exp\left[-\beta_j(\eta - t)/2\right]d\eta & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 < \beta_j^2/4, \\ \dfrac{2}{\zeta_j}\int_0^t W_j(\eta)\sin\left[\zeta_j(t-\eta)/2\right]\exp\left[-\beta_j(\eta - t)/2\right]d\eta & \text{for } \mathrm{Fo}_{q,j}\vartheta_j^2 > \beta_j^2/4; \end{cases}$$

$d_l = \mathrm{Bi}_{x,j}K_{x,j}(\mu_{n,j})\sum_{k=0}^{k_y} g_{k,l}S_k(\gamma_{m,j}) + \overline{K_{x,j}}\left[\mathrm{Bi}_{y=0,j}g_{0,l} + \mathrm{Bi}_{y=1,j}K_{y,j}(\gamma_{m,j})\sum_{k=0}^{k_y} g_{k,l}\right]\Big/ L^2$; $a_2 = (\beta_j + \zeta_j)/2$;

$\overline{K_{x,j}} = \int_0^1 K_{x,j}(\mu_{n,j}X)\,dX$; $\overline{K_{y,j}} = \int_0^1 K_{y,j}(\gamma_{m,j}Y)\,dY$; $S_k(\gamma_{m,j}) = \int_0^1 Y^k K_{y,j}(\gamma_{m,j}Y)\,dY$; $a_1 = (\beta_j - \zeta_j)/2$;

$K_{x,j}(\mu_{n,j}X) = \cos(\mu_{n,j}X)$ and $K_{y,j}(\gamma_{m,j}Y) = \cos(\gamma_{m,j}Y) + \mathrm{Bi}_{y=0,j}\sin(\gamma_{m,j}Y)/\gamma_{m,j}$ are the transformation kernels; $\mu_{n,j}$ (n = 1, 2, …) and $\gamma_{m,j}$ (m = 1, 2, …) are the the roots of the characteristic Eq.: $\mathrm{Bi}_{x,j} = \mu_{n,j}\tan\mu_{n,j}$, $\mathrm{tg}(\gamma_{m,j})\left(1 - \mathrm{Bi}_{y=0,j}\mathrm{Bi}_{y=1,j}/\gamma_{m,j}^2\right) = \left(\mathrm{Bi}_{y=0,j} + \mathrm{Bi}_{y=1,j}\right)/\gamma_{m,j}$.

The obtained solution is characterized by inhomogeneous convergence of Fourier series, because of which, to obtain the result of an acceptable accuracy, it is necessary to increase the count of Fourier series terms for the longitudinal coordinate in solution (10) for large Bio numbers to several hundred or more. The application of the method for improving the convergence, described for the one-dimensional problem in [12], and for the two-dimensional — in the monograph [13], made it possible to obtain the final solution

$$\theta_j(X,Y,t) = \theta_{f,j}(Y,t) + \sum_{n=1}^{\infty} A_{n,j}K_x(\mu_{n,j}X)\sum_{m=1}^{\infty} A_{m,j}K_y(\gamma_{m,j}Y)\left[\theta_{L,j}(\mu_{n,j},\gamma_{m,j},t) - W_j(t)/\vartheta_j^2\right], \tag{11}$$

where constants $C_{1,j}$ and $C_{2,j}$ are determined from the initial conditions (5). Due to the improvement of convergence, it was possible to reduce the number of the series terms by an order of magnitude or more without loss of the solution accuracy.

The reliability of analytic solutions of heat conduction hyperbolic boundary-value problems can be judged from the results of a comparison of the analytic solution of a hyperbolic boundary-value problem for a one-dimensional body (infinite cylinder) with $\mathrm{Bi_x} = 1$, $\mathrm{Fo_q} = 10$, and a special case of equation (3) ($\tau_T = 0$) with a numerical solution of the same problem, given in the monograph [13 , 208-220 pp.] and shown in Fig. 2.

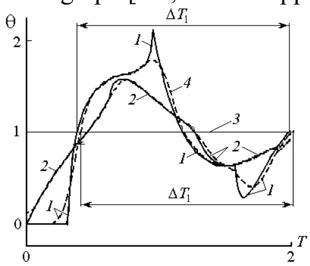

Fig. 2. Transient thermal processes in a cylinder [13]: *1*, *4* - on the axis ($X = 0$); *2* - on the surface ($X = 1$); *3* is the temperature of the medium; *1, 2* - analytical solution; *4* - numerical solution.

The graphs in Fig. 2 obviously show that the analytically calculated transient process more closely reflects the harmonics of high (second, third, etc.) orders than calculated by the numerical method, in spite of the fact that in the second case was used a difference scheme of the second order accuracy, both in coordinate and in time.

3 Conclusions

An analytic description of the heat transfer process in a two-dimensional plate was obtained for the first time according to the Maxwell-Cattaneo-Luikov hypothesis, taking into account the dependences of the heat transfer coefficient of the body surface on time and the temperature of the ambient versus time and the longitudinal coordinate.

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