Improved State Feedback H∞ Control for Flexible Air-Breathing Hypersonic Vehicles on LMI Approach

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Abstract. Focusing on a nonlinear longitudinal dynamical model for Air-breathing Hypersonic Flight Vehicles (AHFV), a linearized model on a nominal trim condition is proposed. To stabilize the flight of an AHFV in the presence of external disturbances and actuator uncertainties, a state feedback H_∞ control is designed. With bounds on the uncertainties, a feedback stabilization problem is converted to an optimal control problem and the cost function is minimized by solving a set of linear matrix inequalities. Since uncertainties in the design of AHFV are inevitable, to make a comparison, a general H_∞ robust controller is constructed by only considering the disturbances firstly. Then the results are extended by incorporating the actual existing uncertainties as well as the external disturbances in the AHFV system. Numerical simulation shows that the controller, which takes both disturbances and uncertainties into account, can effectively stabilize the AHFV system.

1 Introduction

Considering the high demand for extremely fast global strike capability and development of defense systems, the research on hypersonic vehicles has been of a great importance for many researchers. Since the 1970s, the hypersonic speed has generally been defined as at least the value of Mach 5 (five times the speed of sound). High velocity may cause a hypersonic vehicle to be highly sensitive to the change of flight conditions [1], such as the Mach number, and the angle of attack. A comparison among subsonic, supersonic and hypersonic speeds shows that the drag and lift forces become non-linear functions of the angle of attack at hypersonic speeds [1]. The loss of linearity of some parameters may lead the system to be less robust and unstable. An Air-breathing Hypersonic Flight Vehicle (AHFV), such as X-43A, usually is equipped with an air breathing engine which can scoop oxygen from the air. Because of the highly integrated design between airframe and engine, the dynamic characteristics of hypersonic vehicles change drastically with the operating conditions and mass distribution. Meanwhile, the inaccuracy of a mathematical model or control law may also add some uncertainties. In addition, the disturbances from outside atmosphere and density variation at a high Mach number also add uncertainties in the performance of air-breathing propulsion systems [2]. Therefore, a design of a robust and performance oriented control system is the key issue in the design of an Air-breathing Hypersonic Flight Vehicle (AHFV).

Many strategies related to efficient controllers for the hypersonic vehicles have been presented, including adaptive control [3] [4], sliding mode control [5] [6], and H_∞ control [7] [8]. Considering uncertainties due to both aerodynamic and center of gravity, adaptive controllers are suitable for command tracking [9]. However, for an unknown input matrix, the results of the adaptive control have only local stability. The sliding mode control can preferably solve the model uncertain problem [5], and it is insensitive to the out-side interference [6]. But more control force is needed in this method. Due to high parameter perturbation of controller, the air-breathing hypersonic flight model is extremely complex. Therefore, although the single H_∞ controller or μ −synthesis methods may achieve the desired performance [10] [11] [12] [13] [14], it is not easy for these controllers to guarantee the stability and desired performance.

In this paper, the problem of stability is solved by designing an improved state feedback H_∞ control law for a rigid body AHFV. Therefore, the stabilization design problem of a nonlinear system is converted into a problem of optimal control. Since uncertainties are inevitable, to make a comparison, a general controller is constructed by only considering the disturbances firstly. Then the results are extended by incorporating the actual existing uncertainties as well as the external disturbances. Ideally, the control model is MATLAB-based and the simulation results show that the control law is effective and can facilitate the development of the control system.

This paper is organized as follows: Section II presents a linearized AHFV dynamical model. In Section III, by considering only disturbances, a general H_∞ robust controller is presented. Moreover, this control law is further developed by taking both disturbances and uncertainties into account. The simulation results are shown in Section IV, which demonstrates the effectiveness of the proposed control law. Section V draws conclusions.

Notation: The notation used in this paper is fairly standard. The superscript 'T' indicates matrix transposition; $\mathbb{R}$ denotes the *n*-dimensional Euclidean space; sym(P) denotes $P + P^T$); The asterisk (*) is used to represent a term that is introduced by symmetry. For a real symmetric matrix P , $P > 0$ means P is positive definite and $P < 0$ means P is negative definite.

2 Longitudinal dynamical model of hypersonic flight

In this paper, the longitudinal model of an AHFV is adopted from [15]. The aerodynamic coefficients are the typical cruising flight coefficients with Mach number 15. Flight velocity, altitude, path angle and the pitch rate are respectively 15060 ft/s, 110000 ft, 0 deg, 0 deg/s governed by the following equations:

$$\dot{V} = (T\cos\alpha - D)/m - \mu\sin\theta / r^2$$
$$\dot{\theta} = (L + T\sin\alpha)/(mV) - [(\mu - V^2 r)\cos\theta]/(Vr^2)$$
$$\dot{h} = V\sin\theta$$
$$\dot{\alpha} = q - \dot{\theta}$$
$$\dot{q} = M_{yy}/I_{yy}$$

Where V(ft/s) , θ(deg) , h(ft) , α(deg) , q(deg/s) , m(slug) , $\mu(\mathrm{ft^3/s^2})$, and $I_{yy}(\mathrm{slug\cdot ft^2})$ denote velocity, flight-path angle, altitude, angle of attack, pitch rate, mass, gravitational constant, and moment of inertia, respectively. The equations for lift $L(1\mathrm{bf})$, drag $D(1\mathrm{bf})$, thrust $T(1\mathrm{bf})$, pitching moment $M_{yy}(1\mathrm{bf}\cdot\mathrm{ft})$, and the radial distance from earth's center r(ft) are:

$$L = \frac{1}{2}\rho V^2 S C_L$$
$$D = \frac{1}{2}\rho V S^2 C_D$$
$$T = \frac{1}{2}\rho V S^2 C_T$$
$$r = h + R_E$$
$$M_{yy} = \frac{1}{2}\rho V^2 S\bar{c}[C_M(\alpha) + C_M(\delta_e) + C_M(q)]$$

with the following lift coefficient C_L , drag coefficient C_D , thrust coefficient C_L ,and pitching moment coefficients C_α , C_{δ_e} , C_q :

$$C_L = \alpha(0.493 + 1.91/M)$$
$$C_D = 0.0082(171\alpha^2 + 1.15\alpha + 1) \times (0.0012M^2 - 0.054M + 1)$$
$$C_T = \begin{cases} C_{*T}(1+0.15)\beta & \text{if} \beta < 1 \\ C_{*T}(1+0.15\beta) & \text{if} \beta \geq 1 \end{cases}$$
$$C_M(\alpha) = 10^{-4}(0.06 - e^{-M/3}) \times (-6565\alpha^2 + 6875\alpha + 1)$$
$$C_M(\delta_e) = 0.0292(\delta_e - \alpha)$$
$$C_M(q) = \frac{q\bar{c}}{2V}(-0.025M + 1.37) \times (-6.83\alpha^2 + 0.303\alpha - 0.23)$$

where

$$C_{*T} = 0.0105[1 - 164(\alpha - \alpha_0)^2](1 + 17/M)$$

M is the Mach number,

$$M = V/a \quad (a = 8.99\times10^{-9}h^2 - 9.16\times10^{-4}h + 996)$$

and ρ denotes the air density $\rho = 2.38\times10^{-3-(h/24000)}$.

The longitudinal model of the hypersonic vehicle can be transformed into a general MIMO nonlinear system form [16]:

$$\dot{X} = f(X) + \sum_{k=1}^{2} g_k(X)U_k = F(X,U)$$

where the state and control vectors are $X = [V, \theta, h, \alpha, q]^{\mathrm{T}}$ and $U = [\beta, \delta_e]^{\mathrm{T}}$.

In order to simplify the design of controller for the longitudinal model, the model is linearized which is represented by:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 u(t) \\ z(t) = Cx(t) \end{cases}$$

$$A = \frac{\partial F(X,U)}{\partial X}\Big|_{X=X_0, U=U_0} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ -a_{21} & -a_{22} & -a_{23} & -a_{24} & 1 \\ a_{51} & 0 & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

$$B_1 = \frac{\partial F(X,U)}{\partial U}\Big|_{X=X_0, U=U_0} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \\ 0 & 0 \\ -b_{21} & 0 \\ 0 & b_{52} \end{bmatrix}.$$

and

$$a_{11} = \frac{\rho S}{2m}\left[(2VC_T + V^2\frac{\partial C_T}{\partial V})\cos\alpha - 2VC_D - V^2\frac{\partial C_D}{\partial V}\right],$$
$$a_{12} = \frac{-\mu\cos\theta}{r^2},$$
$$a_{13} = -\frac{\rho V^2 S}{48000m}(C_T\cos\alpha - C_D) + \frac{2\mu\sin\theta}{r^3},$$
$$a_{14} = \frac{\rho V^2 S}{2m}(1 + \frac{17}{M})(1+0.15)\beta \times 0.0105\{-328(\alpha - \alpha_0) \times \cos\alpha - [1 - 164(\alpha-\alpha_0)^2]\sin\alpha\} - \frac{\rho V^2 S}{2m} \times 0.0082(0.0012M^2 - 0.054M + 1)(342\alpha + 1.15),$$

$$a_{21}=\frac{-1}{mV^2}(L+T\sin\alpha)+\frac{1}{mV}(\rho VSC_L+\rho VSC_T\sin\alpha)$$
$$+\frac{\mu\cos\theta}{V^2r^2}+\frac{\cos\theta}{r},$$
$$a_{22}=\frac{\mu\sin\theta}{Vr^2}-\frac{V\sin\theta}{r},$$
$$a_{23}=-\frac{\rho VS}{48000m}(C_L+C_T\sin\alpha)+\frac{2\mu\cos\theta}{Vr^3}-\frac{V\cos\theta}{r^2},$$
$$a_{24}=\frac{1}{2mV}\left[\rho V^2S(0.493+\frac{1.19}{M})+3.9606\rho V^2S\right.$$
$$\left.\times(1+\frac{17}{M})\beta(\alpha-\alpha_0)+2T\cos\alpha\right],$$
$$a_{31}=\sin\theta,\quad a_{32}=V\cos\theta,\quad a_{51}=\frac{1}{I_{yy}}M_{yy_v},$$
$$a_{53}=\frac{1}{I_{yy}}M_{yy_h},\quad a_{54}=\frac{1}{I_{yy}}M_{yy_\alpha},\quad a_{55}=\frac{1}{I_{yy}}M_{yy_q},$$
$$M_{yy_v}=\rho VS\overline{c}\times[C_M(\alpha)+C_M(\delta_e)+C_M(q)]$$
$$+\frac{1}{2}\rho V^2S\overline{c}\left(\frac{\partial C_M(\alpha)}{\partial V}+\frac{\partial C_M(\delta_e)}{\partial V}+\frac{\partial C_M(q)}{\partial V}\right),$$
$$M_{yy_h}=\frac{1}{2}\frac{\partial\rho}{\partial h}V^2S\overline{c}[C_M(\alpha)+C_M(\delta_e)+C_M(q)]$$
$$+\frac{1}{2}\rho V^2S\overline{c}\left(\frac{\partial C_M(\alpha)}{\partial M}\frac{\partial M}{\partial h}+\frac{\partial C_M(\delta_e)}{\partial M}\frac{\partial M}{\partial h}\right.$$
$$\left.+\frac{\partial C_M(q)}{\partial M}\frac{\partial M}{\partial h}\right),$$
$$M_{yy_\alpha}=\frac{1}{2}\rho V^2S\overline{c}[\frac{\partial C_M(\alpha)}{\partial\alpha}+\frac{\partial C_M(\delta_e)}{\partial\alpha}+\frac{\partial C_M(q)}{\partial\alpha}]$$
$$M_{yy_q}=\frac{1}{4}\rho VS\overline{c}^2(-0.025M+1.37)$$
$$\times(-6.83\alpha^2+0.303\alpha-0.23),$$
$$b_{11}=\frac{1}{m}T_\beta\cos\alpha,\quad b_{21}=\frac{1}{mV}T_\beta\sin\alpha,\ b_{52}=\frac{1}{I_{yy}}M_{yy\delta},$$
$$T_\beta=0.0060375\rho V^2S[1-164(\alpha-\alpha_0)^2](1+\frac{17}{M}),$$
$$M_{yy\delta}=0.0146\rho V^2S\overline{c}.$$

3 An improved state feedback H_∞ controller design

In this work, the control objective is to robustly stabilize the flight of AHFV in the presence of disturbances and uncertainties. To transform a feedback stabilizability problem into an optimal control problem, an improved state feedback H_∞ control is proposed by considering the Linear-Quadratic Regulator (LQR) control, which involves finding a controller to minimize the value of following cost function:

$$J_L=\int_0^\infty[u^T(t)Ru(t)+x^T(t)Qx(t)]dt,$$

where $R\in\mathbb{R}^{2\times2}$ and $Q\in\mathbb{Q}^{5\times5}$ are given positive-definite symmetric matrices.

The conventional LQR control does not consider disturbances which may significantly compromise the stability of system. In order to increase the robustness, the external disturbances $\omega(t)$ should be taken into account. For a guaranteed cost controller $u(t)=Kx(t)$, a system with disturbances can be expressed as follows:

$$\begin{cases}\dot{x}(t)=(A+B_1K)x(t)+B_2\omega(t),\\ z(t)=C_1x(t),\end{cases}\qquad(1)$$

where B_2 is a constant matrix and $z(t)$ is the controlled output. The objective is to find a desired controller $u(t)=Kx(t)$ to make the system (1) asymptotically stable and minimize the cost function. Hence, the analysis of stability performance is required.

Theorem 1 Consider the AHFV dynamics (1) with given parameters. The system is asymptotically stable with a given H_∞ performance if there exists a positive definite sym-metric matrix $P=P^{\mathrm{T}}>0$ such that the following condition holds:

$$\begin{bmatrix}\mathrm{sym}(P(A+B_1K)) & PB_2 & \begin{bmatrix}Q^{\frac{1}{2}} & 0\end{bmatrix}+K^{\mathrm{T}}\begin{bmatrix}0 & R^{\frac{1}{2}}\end{bmatrix}\\ * & -\gamma^2I & 0\\ * & -\gamma^2I & -I\end{bmatrix}<0\qquad(2)$$

Proof. Define a Lyapunov function as

$$V(t)=x^{\mathrm{T}}(t)Px(t),\qquad(3)$$

where P is a positive-definite symmetric matrix. The H_∞ cost function is as follows:

$$J_H=\int_0^\infty[z^{\mathrm{T}}(t)z(t)-\gamma^2\omega^{\mathrm{T}}(t)\omega(t)+\dot{V}]dt,$$

where $\dot{V}$ is the derivative of $V(t)$.

In order to connect the H_∞ performance index with LQR control, a new vector is defined:

$$z(t)=\begin{bmatrix}Q^{\frac{1}{2}} & 0\end{bmatrix}^{\mathrm{T}}x(t)+\begin{bmatrix}0 & R^{\frac{1}{2}}\end{bmatrix}^{\mathrm{T}}u(t).$$

It can be proved that

$$z^{\mathrm{T}}(t)z(t)=u^{\mathrm{T}}(t)Ru(t)+x^{\mathrm{T}}(t)Qx(t)$$

Thus, by considering the disturbances under the zero initial condition, the improved LQR cost function with a prescribed H_∞ performance can be rewritten as

$$J=\int_0^\infty[u^{\mathrm{T}}(t)Ru(t)+x^{\mathrm{T}}(t)Qx(t)-\gamma^2\omega^{\mathrm{T}}(t)\omega(t)+\dot{V}]dt.$$

The integrand of cost function is denoted as J_*

$$J_*=z^{\mathrm{T}}(t)z(t)-\gamma^2\omega^{\mathrm{T}}(t)\omega(t)+\dot{V},$$
$$=\begin{bmatrix}x(t)\\ \omega(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathrm{sym}(P(A+B_1K))+C_1^{\mathrm{T}}C_1 & PB_2\\ B_2^TP & -\gamma^2I\end{bmatrix}\begin{bmatrix}x(t)\\ \omega(t)\end{bmatrix},$$

where

$$C_1=\begin{bmatrix}Q^{\frac{1}{2}} & 0\end{bmatrix}^{\mathrm{T}}+K^{\mathrm{T}}\begin{bmatrix}0 & R^{\frac{1}{2}}\end{bmatrix}^{\mathrm{T}}.$$

According to the Schur complement, the conditional in equality can be replaced by (23):

$$\begin{bmatrix} \mathrm{sym}(P(A+B_1K)) & PB_2 \\ B_2^T P & -\gamma^2 I \end{bmatrix} - \begin{bmatrix} C_1^{\mathrm{T}} \\ 0 \end{bmatrix} [-I] \begin{bmatrix} C_1 & 0 \end{bmatrix} < 0.$$

Then

$$\begin{bmatrix} \mathrm{sym}(P(A+B_1K)) + C_1^T C_1 & PB_2 \\ B_2^{\mathrm{T}} P & -\gamma^2 I \end{bmatrix} < 0$$

Therefore:

$$J_* < 0,$$

and

$$\begin{aligned} &\mathrm{sym}(P(A+B_1K)) + C_1^T C_1 < 0, \\ &\mathrm{sym}(P(A+B_1K)) < 0. \end{aligned} \tag{4}$$

The quadratic Lyapunov function candidate is shown as (3) and taking the time derivative of V (t) will result:

$$\dot{V}(t) = x^{\mathrm{T}}(t)\mathrm{sym}(P(A+B_1K))x(t).$$

Thus $\dot{V} < 0$, which means the system is stable.

On the other hand, since $J_* < 0$, the following inequality holds:

$$\int_0^\infty \dot{V} dt + \int_0^\infty (z^{\mathrm{T}}(t)z(t) - \gamma^2 \omega^{\mathrm{T}}(t)\omega(t))dt < 0,$$

Assuming zero condition, integration from 0 to ∞ yields:

$$\| z(t) \|_2^2 < \gamma^2 \| \omega(t) \|_2^2 .$$

Therefore, the H_∞ performance is guaranteed. The proof is completed.

Now, the minimum optimal problem is converted into a problem of solving a prescribed H_∞ control, which will significantly enhance the robustness of system. In this control problem, transfer function from disturbances $\omega(t)$ to output $z(t)$ would satisfy the H_∞ norm bound restriction. At the same time, by substituting the control input $u(t)$ into (1), the control system is turned into:

$$\begin{cases} \dot{x}(t) = (A+B_1K)x(t) + B_2\omega(t), \\ z(t) = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix}^{\mathrm{T}} x(t) + K^{\mathrm{T}} \begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix}^{\mathrm{T}} x(t). \end{cases}$$

Furthermore, in order to obtain desired dynamics of the closed-loop systems, usually some pole placement constraints are considered. In this paper a disc pole placement region is chosen [17].

Lemma 1 (Disc region) Let $\mho(\eta, r)$ denotes any disc region centered in η with radius r in a complex plane ($\eta, r \in \mathbb{R}$ and $r > 0$). Then all the eigenvalues of A in (1) lie in the region $\mho(\eta, r)$ if and only if there exists a matrix $P > 0$ satisfying

$$\begin{bmatrix} -P & P(\bar{A} - \eta I) \\ * & -r^2 P \end{bmatrix} < 0. \tag{6}$$

where $\bar{A} = A + B_1 K$.

Now, a multi-objective controller can be obtained by setting $P = X^{-1}$.

Theorem 2 Consider the system in (1). If the scalars r and η are known and there exist matrices $X > 0$ and Y with appropriate dimensions and also the following LMIs are satisfied

$$\begin{bmatrix} AX + XA^{\mathrm{T}} + B_1 Y + Y^{\mathrm{T}} B_1^{\mathrm{T}} & B_2 & X\begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix} + Y^{\mathrm{T}} \begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix} \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \tag{7}$$

and

$$\begin{bmatrix} -X & AX + B_1 Y - \eta X \\ * & -r^2 X \end{bmatrix} < 0, \tag{8}$$

Then there is a proper controller K which can make system described by (1) asymptotically stable and the poles lie in a disc region. The nominal controller gain matrix is given by:

$$K = YX^{-1}. \tag{9}$$

Proof. In order to get the state feedback controller K, congruence transformation is performed on (4) by P^{-1}

$$P^{-1}[P(A+B_1K) + (A+B_1K)^{\mathrm{T}} P^{\mathrm{T}}]P^{-1} < 0.$$

Accordingly,

$$(A+B_1K)P^{-1} + P^{-1}(A+B_1K)^{\mathrm{T}} < 0.$$

Let $X = P^{-1}$ ($X = X^{\mathrm{T}} > 0$),therefore:

$$AX + B_1KX + XA^{\mathrm{T}} + XK^{\mathrm{T}} B_1^{\mathrm{T}} < 0.$$

To calculate the inequality, it is denoted that $Y = KX$, thus

$$AX + B_1 Y + XA^{\mathrm{T}} + Y^{\mathrm{T}} B_1^{\mathrm{T}} < 0,$$

and the controller is:

$$K = YX^{-1}.$$

When $K = YX^{-1}$, performing the congruence transformation to (7) by diag $\{P, I, I\}$, condition (2) is obtained. On the other hand, performing congruence transformation to (8) by diag $\{P, P\}$, one can get (6). The proof is finished.

Remark 1 In the problem formulation above, only external disturbances are considered. However, the actuator uncertainties are inevitable in the practical control system. Therefore, it is quite necessary to incorporate both external disturbances and actuator uncertainties factors to the system model.

A controller is designed to make AHFV asymptotically stable with a given H_∞ attenuation level γ considering external disturbances as well as actuator uncertainties. Assuming the actuator uncertainties can be denoted by ΔA and actual system matrix $\bar{A} = A + \Delta A$, the AHFV dynamical system can be written as

$$\begin{cases} \dot{x}(t) = (A+\Delta A)x(t) + B_1 u(t) + B_2 \omega(t), \\ z(t) = C_1 x(t). \end{cases} \tag{10}$$

The uncertain matrix ΔA is practically assumed to be norm bounded, which means

$$\Delta A = LN(t)E, \tag{11}$$

where L and E are known constant matrices of certain appropriate dimensions. $N(t)$ is an unknown time-varying variable which satisfies $N^{\mathrm{T}}(t)N(t) \le I$.

In order to deal with the uncertainties, the following lemma is used [18]:

Lemma 2 Let $\Theta = \Theta^T$, $\bar{L}$ and $\bar{E}$ be real matrices with compatible dimensions and $N(t)$ be time-varying variable which satisfies (11). Then the following condition

$$\Theta + \bar{L}N(t)\bar{E} + \bar{E}^{\mathrm{T}}N^{\mathrm{T}}(t)\bar{L}^{\mathrm{T}} < 0, \qquad (12)$$

holds if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \Theta & \varepsilon\bar{L} & \bar{E}^{\mathrm{T}} \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0. \qquad (13)$$

The following theorem is derived to eliminate the influence from the actuator uncertainties.

Theorem 3 Consider the AHFV dynamics in (10) with given parameters. The system is asymptotically stable with a given H_∞ performance γ if there exists a positive-definite symmetric matrix $P = P^{\mathrm{T}}$ such that the following inequality holds

$$\begin{bmatrix} \mathrm{sym}(P(A+B_1K)) & PB_2 & \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix} + K^{\mathrm{T}}\begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix} & PL & \varepsilon E^{\mathrm{T}} \\ * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\varepsilon I & 0 \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (14)$$

Proof. According to Lemma 2, the system described by (10) is asymptotically stable with a given H_∞ performance γ if the condition in (13) holds. Therefore, the condition (14) can be changed into

$$\begin{bmatrix} \mathrm{sym}(P(A+B_1K)) & PB_2 & \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix} + K^{\mathrm{T}}\begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix} \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} + \begin{bmatrix} PL \\ 0 \\ 0 \end{bmatrix} N(t)\begin{bmatrix} E & 0 & 0 \end{bmatrix} + \begin{bmatrix} E^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} N^{\mathrm{T}}(t)\begin{bmatrix} L^{\mathrm{T}}P^{\mathrm{T}} & 0 & 0 \end{bmatrix} < 0. \qquad (15)$$

By means of Lemma 2, the matrix inequality (14) can be obtained. This completes the proof.

Theorem 4 (Disc region) Let $\mho(\eta, r)$ denotes any disc region centered in η with radius r in a complex plane ($\eta, r \in \mathbb{R}$ and $r > 0$). Then all the eigenvalues of $A + \Delta A$ in (10) lie in the region $\mho(\eta, r)$ if and only if there exists a matrix $P > 0$ satisfying

$$\begin{bmatrix} -P & P(\bar{A} - \eta I) & \varepsilon PL & 0 \\ * & -r^2 P & 0 & E^{\mathrm{T}} \\ * & * & -\varepsilon I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0. \qquad (16)$$

Proof. When taking the uncertain matrix ΔA into account, Lemma 1 can be changed into

$$\begin{bmatrix} -P & P(\bar{A} + \Delta A - \eta I) \\ * & -\gamma^2 P \end{bmatrix} < 0, \qquad (17)$$

since $\Delta A = LN(t)E$, (17) can be rewritten as

$$\begin{bmatrix} -P & P(\bar{A} - \eta I) \\ * & -r^2 P \end{bmatrix} + \begin{bmatrix} PL \\ 0 \end{bmatrix} N(t)\begin{bmatrix} 0 & E \end{bmatrix} + \begin{bmatrix} 0 \\ E^T \end{bmatrix} N^{\mathrm{T}}(t)\begin{bmatrix} L^{\mathrm{T}}P^{\mathrm{T}} & 0 \end{bmatrix} < 0. \qquad (18)$$

By means of Lemma 2, the matrix inequality (16) can be obtained. This completes the proof.

Under an uncertain precondition, the H_∞ based LQR control design with pole placement can be presented in the following theorem.

Theorem 5 Consider the system in (10). If the scalars r and η are known and there exist matrices $X > 0$ and Y with appropriate dimensions, the following LMIs are satisfied

$$\begin{bmatrix} AX + XA^{\mathrm{T}} + B_1Y + Y^{\mathrm{T}}B_1^{\mathrm{T}} & B_2 & X\begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix} + Y^{\mathrm{T}}\begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix} & \varepsilon L & XE^{\mathrm{T}} \\ * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \qquad (19)$$

and

$$\begin{bmatrix} -X & AX + B_1Y - \eta X & \varepsilon L & 0 \\ * & -r^2 X & 0 & XE^{\mathrm{T}} \\ * & * & -\varepsilon I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0. \qquad (20)$$

Then there is a proper controller K can make system (10) asymptotically stable with poles lying in a disc region. The nominal controller gain matrix is given by:

$$K = YX^{-1}. \qquad (21)$$

Proof. Applying congruence transformation to (14) by diag $\{P^{-1},I,I,I,I\}$, and set $X=P^{-1},Y=KP^{-1}$, will result:

$$\begin{bmatrix} AX+XA^{\mathrm{T}}+B_1Y+Y^{\mathrm{T}}B_1^{\mathrm{T}} & B_2 & X\begin{bmatrix}Q^{\frac{1}{2}} & 0\end{bmatrix}+Y^{\mathrm{T}}\begin{bmatrix}0 & R^{\frac{1}{2}}\end{bmatrix} & L & X\varepsilon E^{\mathrm{T}} \\ * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\varepsilon I & 0 \\ * & * & * & * & -\varepsilon I \end{bmatrix}<0, \tag{22}$$

Then, left-multiply (22) by diag $\{I,I,I,I,\varepsilon^{-1}\}$ and right-multiply (22) by the transpose of diag $\{I,I,I,I,\varepsilon^{-1}\}$. Next, replace ε^{-1} by ε_1 and multiply the resulting LMI by diag $\{I,I,I,\varepsilon_1,I\}$ on the left and its transpose on the right to obtain (19). Additionally, by application of congruence transformation on (20) using diag $\{P,P,I,I\}$, the condition (16) can be obtained.

Theorem 5 indicates that even in the presence of unavoidable actuator uncertainties, the robustness of the real system can be guaranteed by computing the inequalities.

4 Simulation results

To illustrate the performance of improved state feedback H_∞ controller, some numerical simulations are done using MATLAB. The parameters used in this study are adopted from [15] as listed in Table 1.

Table 1. Trim Condition States Value.

Symbol	Value	Symbol	Value
h	110000ft	θ	0rad
S	3603ft^2	β	0.183
$\bar{c}$	80ft	δ	–0.0066
m	9375slug	V	15060ft/s
I_{yy}	7×10^6slug·ft^2	α	0.0315rad
R_E	20903500ft	M	15
μ	1.39×10^{16} ft$^3/s^2$	q	0rad/s
α_0	0.0315rad		

Under a straight-and-level flight trim condition, the values for state vectors and input control are:

$$x_0=[15060\quad 0\quad 110000\quad 0.0315\quad 0]^{\mathrm{T}},$$

$$u_0=[0.183\quad -0.0066]^{\mathrm{T}}.$$

The values of A, B_1 are shown as follows:

$$A=\begin{bmatrix} -3.4700\times10^{-5} & -31.4788 & -7.2040\times10^{-6} & -47.8373 & 0 \\ 2.7135\times10^{-7} & 0 & -5.3151\times10^{-8} & 0.0406 & 0 \\ 0 & 15060 & 0 & 0 & 0 \\ -2.7135\times10^{-7} & 0 & 5.3151\times10^{-8} & -0.0406 & 1 \\ 5.5375\times10^{-6} & 0 & -1.4010\times10^{-7} & 0.5923 & -0.0682 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 27.2963 & 0 \\ 5.7113\times10^{-5} & 0 \\ 0 & 0 \\ -5.7113\times10^{-5} & 0 \\ 0 & 3.3168 \end{bmatrix}.$$

When there are only disturbances, the gain matrices K_1 can be computed by (9). The external disturbances are modeled by Dryden turbulence model, and the turbulence spectrum is defined by the weight [13]:

$$\omega(s)=\sqrt{\frac{2V_0\sigma_u^2}{L_u}}\frac{1}{s+V_0/L_u},$$

$$\sigma_u=5.8ft/s, L_u=65574ft/s\ .$$

The matrix of external disturbances is $B_2=[0\quad 0.5\quad 0\quad 0.5\quad 0]^{\mathrm{T}}$. Within a desired disc region, assuming $Q=0.0001I$, $R=0.01I$, controller $u(t)=Kx(t)$ is designed to make the system asymptotically stable. It is assumed that the disc region is $\mho(-8,10)$ and $Q=0.0001I$, $R=0.01I$. The gain matrices K_1 is

$$K_1=\begin{bmatrix} -0.5636 & -32.7615 & -0.0089 & 1.6366 & -0.0023 \\ 0.0454 & -2.1398\times10^4 & -4.7814 & -132.1430 & -9.7263 \end{bmatrix} \tag{23}$$

Minimum disturbance attenuation level is $\gamma_1=3.4652$.

The uncertainties inherently exist in the system and will inevitably degrade the performance of AHFV. Hence, it is important to consider these uncertainties which may significantly affect the stability of system.

By using the approach developed in this paper, the gain matrix K_2 can be computed by (21).

$$K_2 = \begin{bmatrix} -0.6245 & 1.8893\times10^3 & -0.1323 & 16.5366 & 0.5268 \\ 0.8663 & -2.0088\times10^4 & -3.1857 & -136.7458 & -1.1669 \end{bmatrix}, \quad (24)$$

The minimum disturbance attenuation level is $\gamma_2 = 5.4144$.

Figures 1 and 2 show the simulation results when uncertainties exist. As shown, the controller developed in [19] cannot stabilize the system while the controller developed in this paper is able to attenuate the uncertainties and can quickly enter the stable state when uncertainties happen.

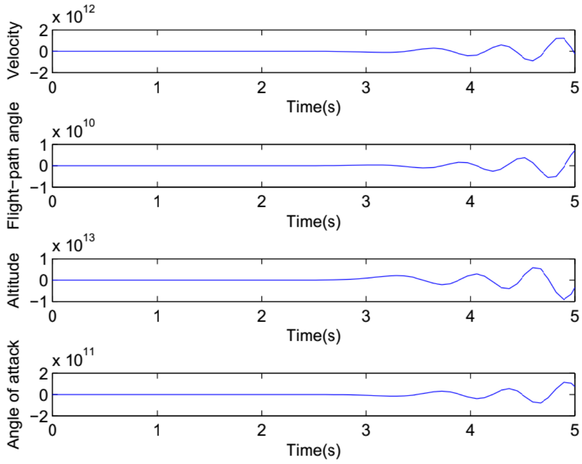

Figure 1. Output responses using the approach developed in [19].

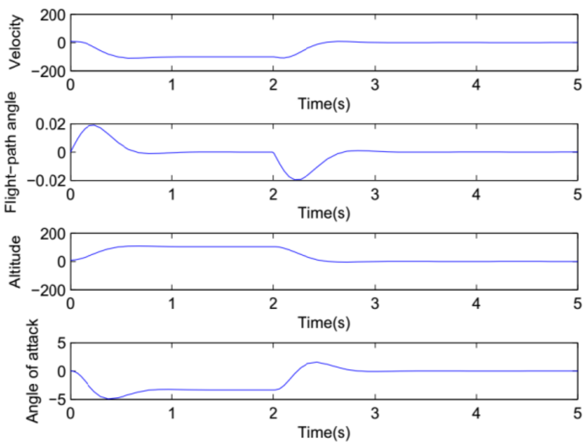

Figure 2. Output responses using the approach developed in this paper.

Remark 2 This paper is motivated by [19]. Gao et al presenting an LQR controller but the actual existing disturbances $\omega(t)$ and uncertainties ΔA are not considered. In this paper, firstly, when there are only disturbances, an improved state feedback H_∞ controller is developed by considering the H_∞ and LQR indexes, which can increase the robustness of system. Secondly, since the actual existing disturbances is inevitable, it is critical to take the actual existing disturbances into account in terms of both theoretical merits and practical application perspective. When there are both disturbances and uncertainties, the output responses in the control mode of [19] cannot converge to an equilibrium state, however, using the control law in this paper, the output responses can rapidly converge to equilibrium states, which means the stability performance of AHFV system is significantly improved.

5 Conclusions

In this paper, a technique of designing improved state feedback H_∞ controller has been developed. Indexes for H_∞ performance, quadratic performance and pole placement are concerned together. By converting the mixed cost function, the multi-objective controller can be deduced by LMIs. Moreover, in terms of a more complete consideration on both external disturbances and actuator uncertainties, the simulation results in [19] and in this paper are presented to make a clear comparison. The results show that, the controller developed in this paper, which takes the actual existing actuator uncertainties into account, can stabilize the flight of AHFV successfully. Performances of flexible air-breathing hypersonic vehicles demonstrate the capability of the algorithm and the feasibility of the controller.

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