Frequency characteristics of a dynamical system at force excitation

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Abstract. Frequency response functions are important characteristics. They define the properties of a dynamic system. Preferably they are used mainly for solving problems in the frequency domain. They are linked to the choice of computational model and to the means of structure excitation. They are complex characteristics, so have their amplitude and phase components. They may be obtained in different ways. Some possibilities of calculating at force excitation are discussed in this paper.

1 Introduction

Dynamic analysis of structures is dependent on the choice of the computational model and the mean of dynamic excitation. In practice the discrete computational model is very often used because the equations of motion have the character of ordinary differential equations. The computational model can be chosen in the spirit of the principles of classical dynamics or in the spirit of the finite element method. Excitation of the structure can be power or kinematic. When choosing a discrete computational model and the force excitation with variable frequency composition it is favourable to use the frequency response functions as characteristics describing the properties of a dynamic system. The paper is devoted to the analysis of such characteristics.

2 Transfer from time to frequency domain

The transfer from time to the frequency domain can be realized by using some integral transformation. The Fourier transform and the Laplace transform are the most commonly used. But also other transformations can be used [1]. For the purpose of this paper the Fourier transform is used. Fourier image of a time function $v(t)$ is denoted as $V(q)$, $V(q) = F\{v(t)\}$, where q is a real number. In our case $q = \omega$ (ω has the meaning of circular frequency in [rad/s]). The complex Fourier transform is defined as

$$V(q) = \int_{-\infty}^{+\infty} v(t) \cdot \mathrm{e}^{-\mathrm{i}q \cdot t} \mathrm{d}t \ . \qquad (1)$$

Inverse Fourier transform is defined as

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$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V(q) \cdot \mathrm{e}^{\mathrm{i}q \cdot t} \mathrm{d}q \,. \tag{2}$$

The function $v(t)$ and its derivations will be transformed as

$$a \cdot v(t) \quad \rightarrow \quad a \cdot V(q)\,,$$

$$\dot{v}(t) \ \text{for} \ v(\pm\infty) = 0 \quad \rightarrow \quad \mathrm{i}q \cdot V(q)\,,$$

$$\ddot{v}(t) \ \text{for} \ v(\pm\infty) = \dot{v}(\pm\infty) = 0 \ \rightarrow \ -q^2 \cdot V(q)\,. \tag{3}$$

3 Definition of frequency response function and its derivation

Dynamical properties of linear system are fully characterized by the response to excitation by simple harmonic functions [2]. Frequency response function $V(p)$ for ($p = \mathrm{i}\omega$) is defined as the ratio of steady state response to harmonic excitation

$$V(\mathrm{i}\omega) = v_{sst} \,/\, F\mathrm{e}^{\mathrm{i}\omega \cdot t}\,. \tag{4}$$

If the entering value is periodical with unit amplitude

$$F(t) = Ff(t) = 1\mathrm{e}^{\mathrm{i}\omega \cdot t} \quad , \tag{5}$$

the exiting value can be written as

$$v(t) = V(\mathrm{i}\omega)\mathrm{e}^{i \cdot \omega \cdot t}\,. \tag{6}$$

Frequency response $V(\mathrm{i}\omega)$ is the complex function and it can be calculated as vector sum of real $\mathrm{Re}[V(\mathrm{i}\omega)]$ and imaginary $\mathrm{Im}[V(\mathrm{i}\omega)]$ parts, Fig. 1.

$$V(\mathrm{i}\omega) = \mathrm{Re}[V(\mathrm{i}\omega)] + \mathrm{Im}[V(\mathrm{i}\omega)] \tag{7}$$

or

$$V(\mathrm{i}\omega) = \left|V(\mathrm{i}\omega)\right| \mathrm{e}^{\mathrm{i}\varphi} \quad , \tag{8}$$

where $\left|V(\mathrm{i}\omega)\right|$ is the absolute value of the frequency response that represents the amplitude of the response $v(t)$ and φ in equation (8) represents the phase of the frequency response.

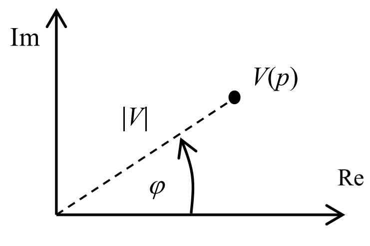

Fig. 1. Frequency response function as a complex function.

When substituting (8) to (6), we obtain

$$v(t) = |V(\mathrm{i}\omega)|\mathrm{e}^{\mathrm{i}\varphi} \cdot \mathrm{e}^{\mathrm{i}\omega \cdot t} = |V(\mathrm{i}\omega)|\mathrm{e}^{\mathrm{i}(\omega \cdot t+\varphi)}. \tag{9}$$

Then we can write

$$|V(\mathrm{i}\omega)| = \sqrt{\mathrm{Re}^2[V(\mathrm{i}\omega)] + \mathrm{Im}^2[V(\mathrm{i}\omega)]} \tag{10}$$

and

$$\varphi = \varphi(\omega) = \mathrm{arctg}(\mathrm{Im}[V(\mathrm{i}\omega)] / \mathrm{Re}[V(\mathrm{i}\omega)]) \ . \tag{11}$$

Now assume discrete computational model with n degrees of freedom (Fig. 2) excited by discrete forces at the points of lumped masses. The equation of motion describing the forced damped oscillations system can be written in the form

$$[m]_D \cdot \{\ddot{v}(t)\} + 2\omega_b [m]_D \cdot \{\dot{v}(t)\} + [k] \cdot \{v(t)\} = \{F(t)\}, \tag{12}$$

where $[m]_D$ is diagonal mass matrix, $[k]$ is stiffness matrix, ω_b is angular damping frequency, $\{v(t)\}$ is the vector of unknown deflections of mass points and $\{F(t)\}$ is the vector of exciting forces. Derivatives with respect to time are denoted by dot over the symbol [3].

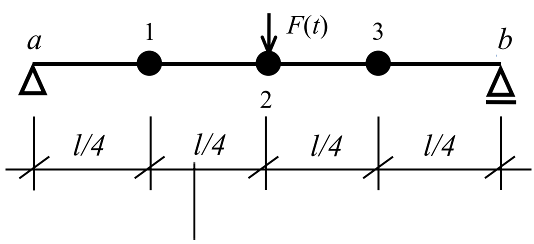

Fig. 2. Computational model of the structure.

Let us apply the Fourier transform to the equation (12). Fourier images of the functions $\{v(t)\}$ and $\{F(t)\}$ are denoted as $\{V(q\}$ and $\{F(q)\}$, where $q = \omega$. Equation (12) is transformed to

$$-q^2[m]_D \cdot \{V(q)\} + iq2\omega_b [m]_D \cdot \{V(q)\} + [k] \cdot \{V(q)\} = \{F(q)\}. \tag{13}$$

Suppose that only k-function of the vector $\{F(q)\}$ is nonzero and all others are zero. It is now possible to define the n^2 frequency responses for $i = 1 \div n$ and $k = 1 \div n$. For the frequency response $\bar{v}_{ik} \equiv \bar{v}_{ik}(q)$ it is valid

$$\bar{v}_{ik} \equiv \bar{v}_{ik}(q) = \frac{V_i(q)}{F_k(q)} \, . \tag{14}$$

This will provide n systems of simultaneous equations, $k = 1 \div n$, to calculate n frequency responses $\bar{v}_{ik}$ at each step of the solution, for $i = 1 \div n$.

$$-q^2 \cdot [m]_D \cdot \{\bar{v}\} + i \cdot q \cdot 2\omega_b \cdot [m]_D \cdot \{\bar{v}\} + [k] \cdot \{\bar{v}\} = \{F_k\},$$

$$([k] - q^2 \cdot [m]_D) \cdot \{\bar{v}\} + i \cdot q \cdot 2\omega_b \cdot [m]_D \cdot \{\bar{v}\} = \{F_k\},$$

$$(([k]-q^2\cdot[m]_D)+i\cdot q\cdot 2\omega_b\cdot[m]_D)\cdot\{\bar{v}\}=\{F_k\},$$

$$(([k]-\omega^2\cdot[m]_D)+i\cdot \omega\cdot 2\omega_b\cdot[m]_D)\cdot\{\bar{v}\}=\{F_k\}. \tag{15}$$

Vector $\{F_k\}$ for the k^{th} system of equations contains zero, only in the k^{th} row is number one.

4 Numerical solution

Numerical calculations were applied to discrete computational model of the real bridge construction with one span made from prefabricated elements I73 with the following parameters: span l = 29.0 m, elastic modulus $E = 3.85 \cdot 10^{10}$ N/m^2, quadratic moment of the cross-section I = 2.391711 m^4, the intensity of mass μ = 19.680 kg/m. Masses of discrete model with 3 degrees of freedom have the following values

$m_1 = m_2 = m_3 = \mu \cdot l/4 = 19680 \cdot 29/4 = 142680$ kg.

Stiffness matrix elements are as follows

$k_{11} = k_{33} = 2.381806419615400 \cdot 10^9$ N/m, $k_{22} = 3.313817627290990 \cdot 10^9$ N/m,

$k_{12} = k_{21} = k_{23} = k_{32} = -2.278249618762560 \cdot 10^9$ N/m,

$k_{13} = k_{31} = 9.320112076755900 \cdot 10^9$ N/m.

Natural frequencies of the model are as follows

$f_{(1)} = 4.0389$ Hz, $f_{(2)} = 16.0432$ Hz, $f_{(1)} = 34.0633$ Hz.

Calculations of frequency response functions have been realized in the frequency range 0 ÷ 40 Hz with a step of 0.01 Hz and are displayed in the form of amplitude and phase characteristics. For excitation at point k = 1 they are shown in Fig. 3, 4, 5 and for excitation at point k = 2 in Fig. 6.

5 Conclusions

Frequency response functions are important characteristics clearly defining the properties of dynamic systems. They are linked to the choice of the computational model and the means of excitation. They can be calculated in various ways, for example by the passage from time to the frequency domain on the basis of the Fourier transform. The usage of derivative procedures can be adopted to solve a wide variety of dynamic tasks [4], [5], [6], [7]. From the above results it can be seen that the values of dominant frequencies are $f_{(1)}$ = 4.04 Hz, $f_{(2)}$ = 16.04 Hz, $f_{(3)}$ = 34.06 Hz. As a result of Maxwell's reciprocity theorem, $\bar{v}_{ik} = \bar{v}_{ki}$. In our specific case, the construction is symmetrical with respect to the axis of symmetry passing through the mass point 2. Due to the symmetry, $\bar{v}_{11} = \bar{v}_{33}$.

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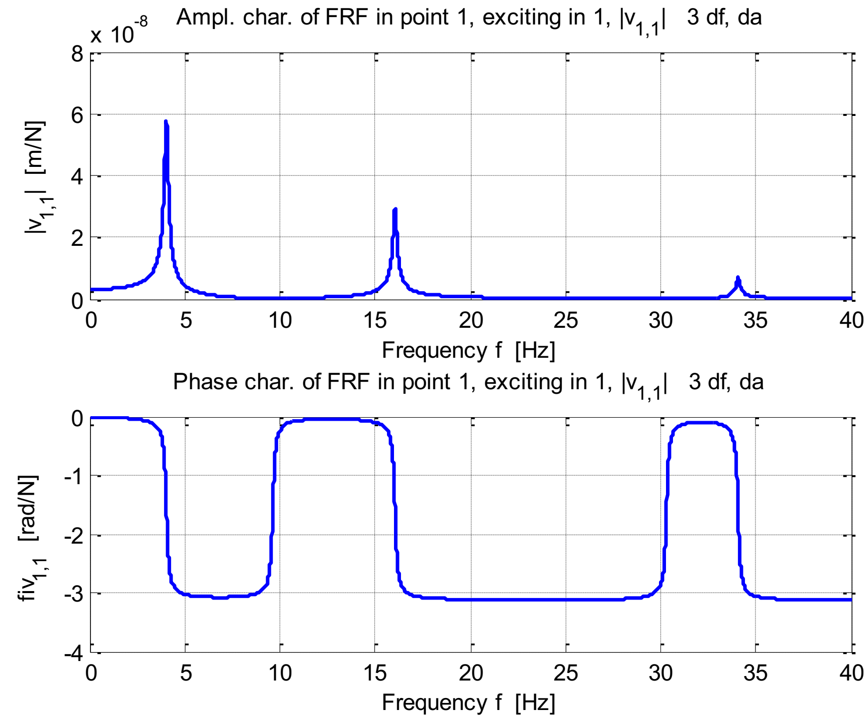

Fig. 3. FRF in point 1, exciting in point 1.

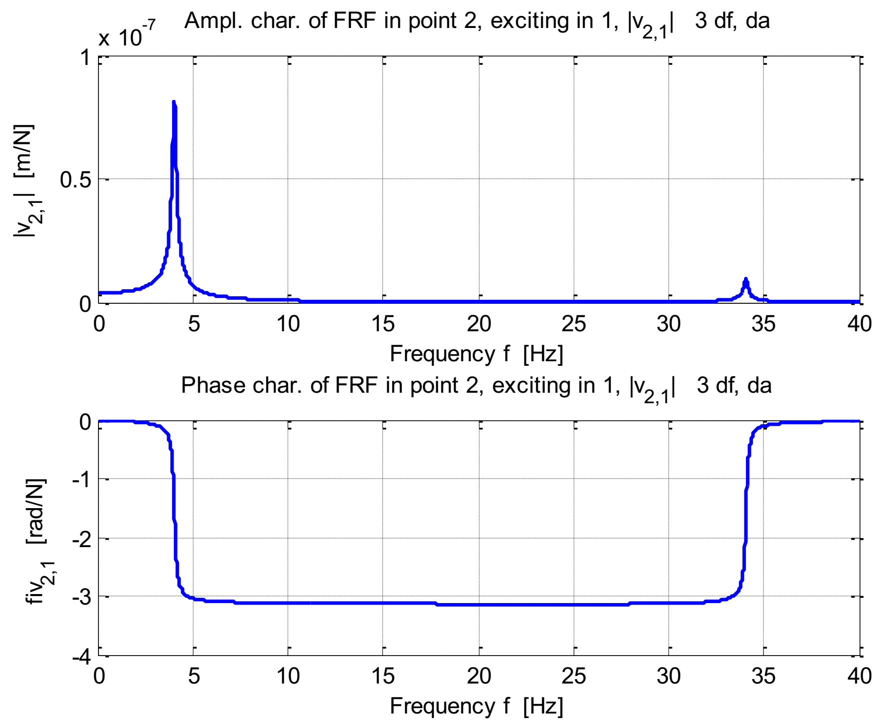

Fig. 4. FRF in point 2, exciting in point 1.

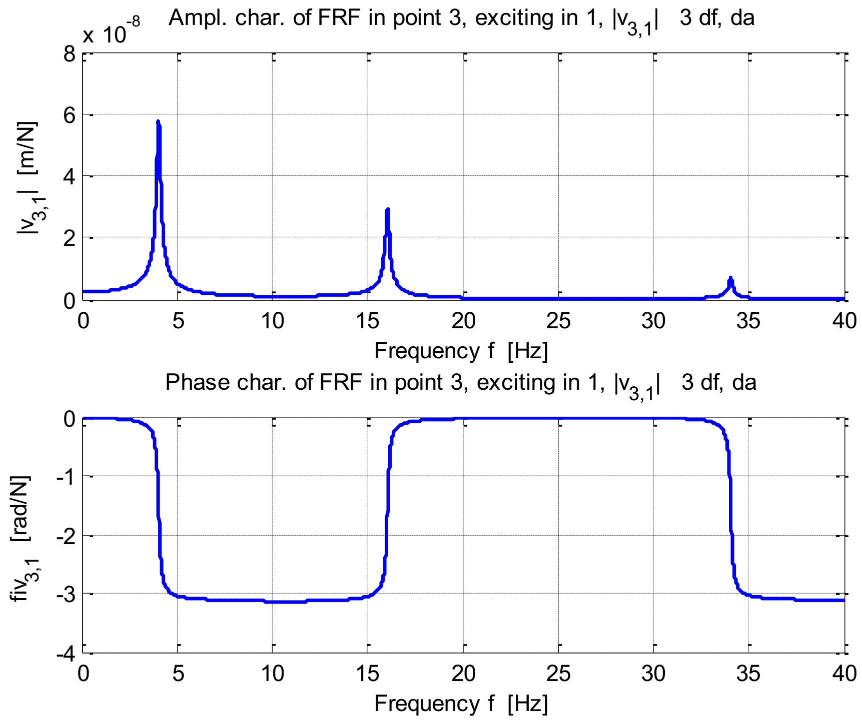

Fig. 5. FRF in point 3, exciting in point 1.

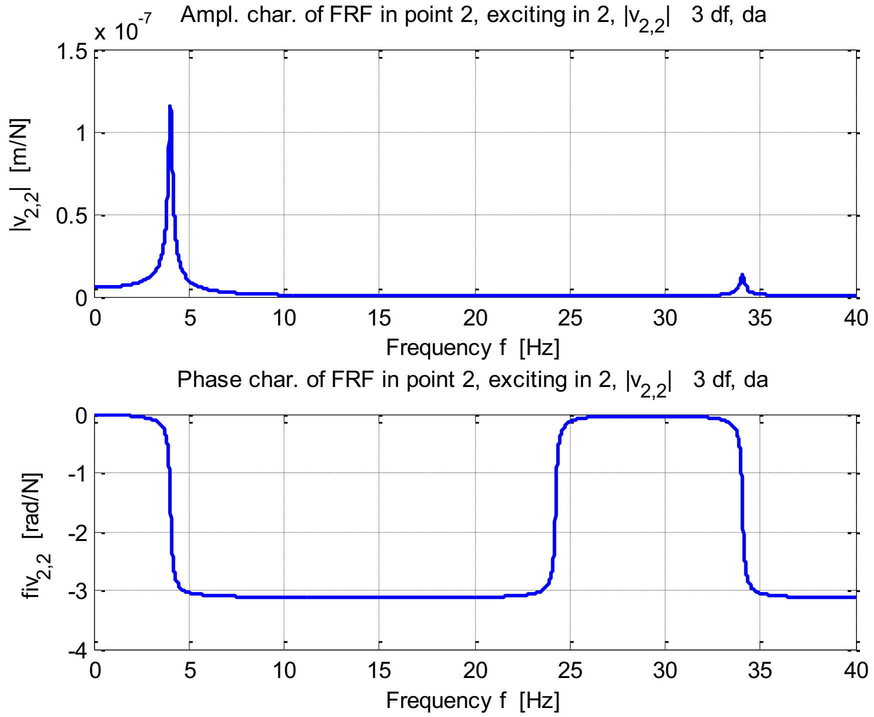

Fig. 6. FRF in point 2, exciting in point 2.

References

1. J. Melcer, et al., *Dynamics of transport structures* (EDIS, Žilina, 2016, in Slovak)
2. M. Baťa, et al., *Dynamics of structures* (SNTL/ALFA, Praha/Bratislava, 1987, in Czech)
3. D. Kuchárová, J. Melcer, *Dynamics of structures* (EDIS, Žilina, 2000, in Slovak)
4. E. Panulinová, Silniční obzor, **62**, 275-279 (2001)
5. K. Kotrasová, E. Kormaníková, International Journal of Mechanics. **10**, 242-252 (2016)
6. M. Mikušová, WIT Trans. Built Env. **134**, 503-514 (2013)
7. I. Olekšáková, O. Ivánková, Adv. Mat. Res. **969**, 148-154 (2014)