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Stability of E. Reyssner's plates on the elastic non-winkler foundation

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Abstract. The problem of E. Reyssner's plate stability lying on an elastic three-dimensional layer with desired elastic constants. The end surfaces of layer are smooth, connection holding. It is assumed that the plate is in a flat stress-strain state of the effects on its cylindrical surface of the self-balanced load, with some numerical parameter characterizing the magnitude of the load at loss of stability of the plate. From the conditions of restraint ties one obtain a system of equations for determining the numerical parameter. Method is given for calculating the lowest value of the parameter at which the plate's loss of stability is fixed. As special cases, the results of the classical theory and model of Winkler foundation are present.

1 Introduction

Let us consider the plate having the shape of Ω and limited by cylindrical contour Γ lying on the elastic layer of the three-dimensional: $-\infty$ $(x, y) < \infty$; $0 \le z_1 \le h_1$. Let the elastic constants be E_1, ν_1 layer, plate thickness h, its elastic constants E, ν, $D = h^3E/12(1-\nu^2)$. The end surfaces of layer are smooth, connection holding. Let us assume that the plate is in a flat stress-strain state of the effects on the contour Γ of self-balanced load $\lambda P_n(s)$, $\lambda P_\tau(s)$, where λ – some numerical parameter, characterizing the magnitude of the load at loss of stability of the plate. From these conditions,

$$\sigma_n\big|_\Gamma = \lambda P_n(S),\ \ \tau_{ns}\big|_\Gamma = -\lambda P_\tau(S),\ \ \tau_{\alpha Z}\big|_{Z=\pm h/2} = 0,\ \ \sigma_Z\big|_{Z=-h/2} = 0; \tag{1}$$

$$\tau_{\alpha Z_1}\big|_{Z_1=0;h_1} = 0,\ \ \omega_1\big|_{Z_1=0} = 0,\ (\alpha = x, y); \tag{2}$$

$$\omega = \omega_1\big|_{Z_1=h_1},\ \ \sigma_Z\big|_{Z=h/2} = \sigma_{Z_1}\big|_{Z_1=h_1}. \tag{3}$$

2 Problem definition

Suppose that for some reason, plate bent a bit. Consider the following conditions of bifurcation of forms of compressed slabs according to Euler equilibrium.

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If the plate is loaded by power system (1), when of its bending compressive forces give the component in the same direction as the transverse load:

$$q = (x, y) = \lambda(\omega_{xx}F_{yy} + \omega_{yy}F_{xx} - 2\omega_{xy}F_{xy}) \tag{4}$$

Where $\Phi(x, y)$ – function of longitudinal forces in the plate, determined by solving biharmonic problem [1]. The relations between the power and the geometric characteristics of the plate of E. Reysner have the form

$$M_\alpha = -D(\partial_\alpha^2 + \nu\partial_\beta^2)\omega + 0,1h^2\partial_\alpha Q_\alpha + 0,1h^2\nu/(1-\nu)(\partial_1 Q_x + \partial_2 Q_y) - \frac{h^2\nu}{12(1-\nu)}q(x, y) \tag{5}$$

$$(\alpha = x, y);$$

$$M_{xy} = -D(1-\nu)\partial_1\partial_2\omega + 0,1h^2(\partial_2 Q_x + \partial_1 Q_y); \tag{6}$$

$$\partial_\alpha M_\alpha - \partial_\beta M_{\alpha\beta} = Q_\alpha, \quad (\alpha = x, y), \quad (\beta = y, x) \tag{7}$$

$$(0,1h^2D^2 - 1)Q_n + 0,1h^2\frac{1}{1-\nu}\partial_\alpha(\partial_1 Q_x + \partial_2 Q_y) = D\alpha D^2\omega + \frac{h^2}{12}\frac{\nu}{1-\nu}\partial_\alpha q(x, y) \tag{8}$$

$$(\alpha = x, y);$$

Stress-strain state of three-dimensional elastic layer is described by the solution of homogeneous equations of the theory of elasticity in displacements [2]:

$$\left\| \begin{matrix} u \\ \nu \end{matrix} \right\| = \cos z_1 D_1 \left\| \begin{matrix} u_0 \\ \nu_0 \end{matrix} \right\| - \frac{x_1}{2}\frac{z_1 \sin z_1 D}{D}\partial_\alpha\theta_0; \tag{9}$$

$$\omega_1 = \frac{\sin z_1 D_1}{D}\omega_0' + \frac{x_1}{2}\left(\frac{\sin z_1 D}{D} - z_1 \cos z_1 D\right)\theta_0; \tag{10}$$

$$\frac{1}{\mu_1}\tau_{\alpha Z_1} = -\frac{\sin z_1 D}{D}\left(D^2\left\| \begin{matrix} u_0 \\ \nu_0 \end{matrix} \right\| - \partial_\alpha\omega_0' - x_1 z_1 \cos z_1 D\partial_\alpha\theta_0\right); \tag{11}$$

$$\frac{1}{\mu_1}\tau_{z_1} = 2\cos z_1 D_1\omega_0' + [x_1 z_1 D \sin z_1 D + (x_1 - 1)\cos z_1 D]\theta_0. \tag{12}$$

Here $z_1 \in [0; h_1]$, $u_0(x, y)$, $\nu_0(x, y)$, $\omega_0'(x, y)$, $\theta_0 = \partial_1 u_0 + \partial_2 \nu_0 + \omega_0'$ – basic unknown functions. Expressions (10) (11) identically satisfy the conditions (2). Other conditions on the ends of the plate and layer allow to solve the problem of the of bifurcation forms of balance of system and to determine the value of the parameter λ. For this purpose substitute (10) – (12) (2) and (3) [3-6]. Get

$$\frac{\sin z_1 D}{D}D^2\left\| \begin{matrix} u_0 \\ \nu_0 \end{matrix} \right\| - \partial_\alpha\omega_0' + x_1 z_1 \cos z_1 D \times \partial_\alpha\theta_0 = 0; \tag{13}$$

$$\omega=\frac{\sin z_1 D}{D}\omega_0'+\frac{x_1}{2}\left(\frac{\sin z_1 D}{D}-h_1\cos h_1 D\right)\theta_0; \tag{14}$$

$$\begin{gathered}\partial_1 Q_x+\partial_2 Q_y+q(x,y)=\mu_1\times\\ \times[x_1 h_1 D\sin h_1 D\theta_0+(x_1-1)\cos h_1 D\theta_0+2\cos h_1 D\omega_0']\end{gathered} \tag{15}$$

Thus, the general solution of the problem of stress-strain state "plate - a layer" system is reduced to solving the system of Equations 13-15 [7-8]. Let us transform it. From (8) and (15) we have

$$\begin{gathered}DD^4\omega=\left(1-\frac{h^2}{60}\frac{12-\nu}{1-\nu}D^2\right)q(x,y)-\mu_1\left(1-0,1h^2\frac{2-\nu}{1-\nu}D^2\right)\times\\ \times\{2\cos h_1 D\omega_0'+[x_1 h_1 D\sin h_1+(x_1-1)\cos h_1 D]\theta_0\}.\end{gathered} \tag{16}$$

Substituting (14) into (16) leads to the Equation

$$\begin{gathered}\left[DD^3\sin h_1 D+2\mu_1\left(1-0,1h^2\frac{2-\nu}{1-\nu}D^2\right)\cos h_1 D\right]\omega_0+\\ +\left\{D\frac{x_1}{2}\left(\frac{\sin z_1 D}{D}-h_1\cos h_1 D\right)+\mu_1\left(1-0,1h^2\frac{2-\nu}{1-\nu}D^2\right)\times\right.\\ \left.\times[x_1 h_1 D\sin h_1 D+(x_1-1)\cos h_1 D]\theta_0\right\}=\left(1-\frac{h^2}{60}\frac{12-\nu}{1-\nu}D^2\right)q(x,y)\end{gathered} \tag{17}$$

Next, the whole issue is to solve the system of Equations 13 and 17 which can be built by semi-inverse methods. For this we represent the desired solution as the sum of two independent solutions – potential and vortex. It is easy to show that the vortex solution does not solve the problem of stability of the system "plate – layer", and therefore we"ll focus only on potential solutions [9]. Let

$$u_0=L_1\partial_1\Pi,\ v_0=L_1\partial_2\Pi,\ \omega_0'=L_2D^2\Pi, \tag{18}$$

where L_1 и L_2 – some differential operators; P – stress function. Substituting (18) in (13), we obtain the equation

$$[(D\sin h_1 D+x_1 h_1 D^2\cos h_1 D)L_1-(D\sin h_1 D-x_1 h_1 D^2\cos h_1 D)L_2]\Pi=0, \tag{19}$$

which are satisfied with identically for the obvious choice of operators:

$$L_i=\frac{\sin h_1 D}{D}+(-1)^i x_1 h_1\cos h_1 D. \tag{20}$$

The unknown functions take the form

$$\left\|\begin{matrix}u_0\\ v_0\end{matrix}\right\|=\left(\frac{\sin h_1 D}{D}-x_1 h_1\cos h_1 D\right)\left\|\begin{matrix}\partial_1\\ \partial_2\end{matrix}\right\|\Pi; \tag{21}$$

$$\omega_0' = \left(\frac{\sin h_1 D}{D} + x_1 h_1 \cos h_1 D \right) D^2 \Pi; \tag{22}$$

$$\theta_0 = 2D \sin h_1 D\Pi . \tag{23}$$

The expressions for the bending plate we find from expression (14)[10-11]:

$$\omega = (x_1 + 1)\sin^2 h_1 D\Pi. \tag{24}$$

The differential equation for the stress function Π we obtain after substituting (22) – (24) (17):

$$\begin{gathered} DD^4 \sin^2 h_1 D + 2\mu_1 x_1 D^2 \left(1 - 0{,}1h^2 \frac{2-\nu}{1-\nu} D^2 \right)\left(1 + \frac{\sin 2h_1 D}{2h_1 D} \right) \Pi = \\ = \left(1 - \frac{h^2}{60} \frac{12-\nu}{1-\nu} D^2 \right) q(x, y). \end{gathered} \tag{25}$$

Where

$$q(x,y) = \lambda_1 x_1 [\Phi_{yy} \sin^2 h_1 D\partial_1^2 \Pi + \Phi_{,xy} \sin^2 h_1 D\partial_2^2 \Pi - 2\Phi_{,xy} \sin^2 h_1 AD\partial_1 \partial_2 \Pi] \tag{26}$$

and biharmonic function – the solution of the problem of plane stress-strain state of the plate [1,12]. As an example, consider the case of a uniform compression plate. Get

$$\Phi_{,xx} = \Phi_{,yy} = -\frac{P}{h}, \quad \Phi_{xy} = 0 \tag{27}$$

Equations (25) and expression (26) take the form

$$\begin{gathered} DD^i \sin^2 h_1 D\Pi + 2\mu_1 x_1 D^2 \left(1 - \frac{h^2}{10} \frac{2-\nu}{1-\nu} D^2 \right)\left(1 + \frac{\sin 2h_1 D}{2h_1 D} \right) \Pi + \\ + \lambda(1 + x_1) \frac{P}{h} \left(1 - \frac{h^2}{60} \frac{12-\nu}{1-\nu} D^2 \right) D^2 \sin^2 h_1 D\Pi = 0 \end{gathered} \tag{28}$$

$$q(x, y) = -\lambda(1 + x_1) \frac{P}{h} D^2 \sin^2 h_1 D\Pi \tag{29}$$

In order to solve a wide range of engineering problems associated with the problem of buckling of rectangular plates of different purposes, is often used the method by which the plate is given some form of depression. For hinged plates such form is received in the form of

$$\omega = a \sin \alpha x \sin \beta y , \tag{30}$$

and stresses the function Π is determined by the differential equation (24):

$$(x_1 + 1)\sin^2 h_1 D\Pi = a \sin \alpha x \sin \beta y .$$

A particular solution of this equation has the form [2,13]

$$\Pi = \frac{a}{(x_1+1)\sin^2 ih_1\gamma}\sin\alpha x\sin\beta y, \tag{31}$$

where $\gamma^2=\alpha^2+\beta^2$.

After substituting (31) into (28) we obtain the equation for λ:

$$2D\gamma^2 - \frac{2\mu_1 x_1 h_1}{\sin^2 ih_1\gamma}\left(1-0,1h^2\gamma^2\frac{2-\nu}{1-\nu}\right)\left(1+\frac{\sin^2 ih_1\gamma}{2ih_1\gamma}\right) - \lambda\left(1+\frac{h^2}{60}\frac{12-\nu}{1-\nu}D^2\gamma^2\right)=0. \tag{32}$$

The lowest value λ characterizes the magnitude of the load on the contour Γ, at which is fixed the loss of stability [14,15].

3 Results and discussions

Applied theories

1. If E. Reysner plate is simply supported along the contour T, the equilibrium equation of forms can be obtained at $\mu_1\to 0$ and $x_1\to 0$. From (4) and (16) we obtain

$$DD^4\omega = \lambda\left(1-\frac{h^2}{60}\frac{12-\nu}{1-\nu}D^2\right)(\omega_{xx}\Phi_{yy}+\omega_{yy}\Phi_{xx}-2\omega_{xy}\Phi_{xy}) \tag{33}$$

Power characteristics are determined by formulas (5) – (8). Neglecting the term with h_2, we get the results of the classical theory.

2. If the expansions of differential operators retains members with h_1^2, we get the base model of Winkler. In the case of the equation of equilibrium forms a uniform compression plate as follows:

$$DD^4\Pi + 2\frac{\mu_1 x_1}{h_1}\Pi + \frac{1+x_1}{2}\lambda\frac{P}{h}D^2\Pi = 0; \tag{34}$$

$$\omega = (1+x_1)\Pi \tag{35}$$

As an example, consider the size of the plate buckling a × b. Assume that the deflection plates and a function of stress is expressed as

$$\omega = (1+x_1)A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \qquad \Pi = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{36}$$

Substituting (36) in (34), define the parameter λ buckling under consideration:

$$D\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]+2\frac{\mu_1 x_1}{h_1}-\frac{1+x_1}{2}\lambda\frac{P}{h}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]=0. \tag{37}$$

Ratios between the size of the plate are installed using the equilibrium conditions

$$\iint\limits_{s} \sigma_z \, dxdy - 2\left[\int_0^b Q_x \, dy + \int_0^a Q_y \, dx \right]_{\Gamma} = 0 . \tag{38}$$

Because the shear forces $Q_x\big|_{\Gamma}$, $Q_y\big|_{\Gamma}$ and contact stress $\sigma_z\big|_{z=h/2}$ have the form

$$Q_x\big|_{x=0} = 2D\frac{\pi}{\alpha}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] A\sin\frac{\pi y}{b}; \tag{39}$$

$$Q_g\big|_{g=0} = 2D\frac{\pi}{\alpha}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] A\sin\frac{\pi x}{a}; \tag{40}$$

$$\sigma_z\big|_{z=h/2} = -L_1\frac{\mu_1 x_1}{h_1} A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{41}$$

then, after integration (38), we obtain

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{\pi^2}\sqrt{\frac{2\mu_1 x_1}{h_1 D}} . \tag{42}$$

From (37) and (41) one can get a simpler Equation:

$$\lambda = \frac{8\mu_1 x_1}{h_1 \pi^2}\left(\frac{P}{h}\frac{a^2+b^2}{a^2 b^2}\right)^{-1} .$$

4 Conclusion

Thus, E. Reysner theory in the zero approximation leads to results of the classical theory. At the same time fulfill all the plate boundary conditions of equilibrium. This is an advantage of the method.

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