

On Linear Hamiltonian Systems

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Abstract. In this paper we consider the normalization of quadratic Hamiltonian. We get the new method to find the generating function of the canonical transformation. We obtain the solution of the system of matrix equations to find this transformation. The corresponding Hamiltonian matrix has multiply eigenvalues.

1 Introduction

The Hamiltonian systems have an important role in fluid mechanics. The systems are of potential interest for statistical mechanics. Let us consider the quadratic Hamiltonian $H = 1/2 \mathbf{w}^T \Omega \mathbf{w}$. By \mathbf{w} denote vector with dimension $2n$, by Ω denote the symmetric $2n$ -by- $2n$ matrix. We consider the Hamiltonian system of differential equations

$$\frac{d\mathbf{w}}{dt} = J\Omega\mathbf{w}, \quad J = \begin{pmatrix} O_n & E_n \\ -E_n & O_n \end{pmatrix},$$

where E_n is identity matrix. With canonical transformation we find the normal form of Hamiltonian matrix $J\Omega$:

$$I = \begin{pmatrix} U & O_n \\ O_n & -U^T \end{pmatrix}, \quad (1)$$

where submatrix U has Jordan normal form.

In the paper [1] the author considers the normalization of Hamiltonian matrix in the case of nonzero eigenvalues. In the paper [2] the author proves the possibility of normalization of non-singular Hamiltonian matrix. In the papers [3 – 5] the authors obtain other methods of normalization the matrix. We find the canonical transformation with generation function. Hamiltonian matrix has multiply nonzero and zero eigenvalues.

2 System of matrix equations

Let us consider the quadratic Hamiltonian

$$H = 1/2 \mathbf{w}^T \Omega \mathbf{w} = 1/2 \mathbf{x}^T A \mathbf{x} + \mathbf{x}^T B \mathbf{y} + 1/2 \mathbf{y}^T C \mathbf{y}, \quad (2)$$

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where both \mathbf{x} and \mathbf{y} are vectors with dimension n , $\mathbf{w} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$. By A, B, C denote n -by- n real matrices. The matrices A and C are symmetric. The matrix

$$V = J\Omega = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix} \quad (3)$$

is corresponding Hamiltonian matrix. Let \mathbf{q}, \mathbf{p} be vectors of new canonical coordinates with the same dimension n . Let us consider the generation function of the canonical transformation:

$$S(\mathbf{x}, \mathbf{p}) = \frac{1}{2} \mathbf{p}^T K \mathbf{p} + \mathbf{p}^T L \mathbf{x} + \frac{1}{2} \mathbf{x}^T M \mathbf{x}, \quad (4)$$

where the n -by- n matrices K, M are symmetric, n -by- n matrix L is non-singular. Since $\frac{\partial S}{\partial \mathbf{p}} = \mathbf{q}$, $\frac{\partial S}{\partial \mathbf{x}} = \mathbf{y}$, we obtain the system of equations:

$$K\mathbf{p} + L\mathbf{x} = \mathbf{q}, \quad L^T \mathbf{p} + M\mathbf{x} = \mathbf{y}.$$

Therefore, the following equation takes place:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} L^{-1} & L^{-1}K \\ ML^{-1} & L^T - ML^{-1}K \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = T \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \quad (5)$$

We want to find an invertible matrix T such that $T^{-1}VT = I$. From the equations (3), (5) it follows that

$$\begin{aligned} VT &= \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix} \begin{pmatrix} L^{-1} & L^{-1}K \\ ML^{-1} & L^T - ML^{-1}K \end{pmatrix} = \\ &= \begin{pmatrix} B^T L^{-1} + CML^{-1} & -B^T L^{-1}K + CL^T - CML^{-1}K \\ -AL^{-1} - BML^{-1} & AL^{-1}K - BL^T + BML^{-1}K \end{pmatrix} \end{aligned}$$

From the equations (1), (5) it follows that

$$TI = \begin{pmatrix} L^{-1} & L^{-1}K \\ ML^{-1} & L^T - ML^{-1}K \end{pmatrix} \begin{pmatrix} U & O_n \\ O_n & -U^T \end{pmatrix} = \begin{pmatrix} L^{-1}U & L^{-1}KU^T \\ ML^{-1}U & -L^T U^T + ML^{-1}KU^T \end{pmatrix}$$

Since the block matrices are equal we obtain the system of matrix equations:

$$\begin{aligned} B^T L^{-1} + CML^{-1} &= L^{-1}U, \quad -AL^{-1} - BML^{-1} = ML^{-1}U, \\ -B^T L^{-1}K + CL^T - CML^{-1}K &= L^{-1}KU^T, \\ AL^{-1}K - BL^T + BML^{-1}K &= -L^T U^T + ML^{-1}KU^T \end{aligned} \quad (6)$$

It is clear that the fourth equation follows from the first, second and third equations. The equations

$$MB^T + MCM = -A - BM, \quad MCL^T = L^T U^T - BL^T \quad (7)$$

follow from first and second equations. Therefore, we get the equivalent system:

$$\begin{aligned} MCM + MB^T + BM + A &= O, \\ L(B^T + CM)L^{-1} &= U, \\ KU^T + L(B^T + CM)L^{-1}K &= LCL^T \end{aligned}$$

Finally, we get the system of matrix equations:

$$\begin{aligned} MCM + MB^T + BM + A &= O, \\ L(B^T + CM)L^{-1} &= U, \\ KU^T + UK &= LCL^T \end{aligned} \tag{8}$$

In this system the unknown matrices M and K are symmetric, matrix L is non-singular. From the first equation we find the matrix M . From the second equation we find the matrices L and U . From the third equation we find the matrix K . Finally we find the normal form (1) of the Hamiltonian matrix (3). We get the new corresponding Hamiltonian:

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^T U \mathbf{q} \tag{9}$$

Theorem. Suppose the generating function (4) of the canonical transformation has non-singular matrix L . The quadratic Hamiltonian (2) has the normal form (9) with the canonical transformation (5) if and only if the matrices K , L and M of the generating function are the solutions of the system (8).

3 Solution of the system of matrix equations

The first equation of the system (8) is called [5 - 7] matrix Riccati equation:

$$MCM + MB^T + BM + A = O \tag{10}$$

Let us consider methods of solving this equation in various special cases.

Case 1. We substitute matrices $C = E$, $B = O$ into the equation (10). Then we get the equation:

$$M^2 + A = O \tag{11}$$

We want to find the symmetric root of the symmetric matrix. The symmetric matrix A is similar to the diagonal matrix L_A with real eigenvalues:

$$A = F^T L_A F; \quad L_A = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix};$$

where the matrix F is the orthogonal matrix, by λ_i we denote the eigenvalues of the matrix A . Let Λ_A be a diagonal matrix such that $\Lambda_A^2 = -L_A$:

$$\Lambda_A = \begin{pmatrix} \pm\sqrt{-\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \pm\sqrt{-\lambda_n} \end{pmatrix}$$

The diagonal entries of the matrix Λ_A are real or pure imaginary numbers. Therefore we get the symmetric solution of the equation (11):

$$M = F^T \Lambda_A F \tag{12}$$

Case 2. Let C be a symmetric positive-definite matrix, $B = O$. We substitute these matrices into the equation (10). Then we get the following equation:

$$MCM + A = O \tag{13}$$

Let Q be such that $C = QQ^T$. After the change $M_1 = Q^T M Q$, $A_1 = Q^T A Q$ we get the following equation:

$$M_1^2 + A_1 = O; \quad A_1^T = A_1 \tag{14}$$

If the matrix M_1 is the symmetric solution of the equation (14) then the matrix

$$M = (Q^T)^{-1} M_1 Q^{-1} \tag{15}$$

is the symmetric solution of the equation (13).

Case 3. Let us substitute matrices $C = E$, $B = B^T$ into the equation (10). Then we get the equation:

$$M^2 + MB + BM + A = O \tag{16}$$

After the change $M = N - B$, $D = B^2 - A$ we get the equation $N^2 = D$. Since matrix D is symmetric matrix we obtain the same equation as the equation (11). Let the matrix F be the orthogonal matrix such that $D = F^T L_D F$, where the matrix L_D is the diagonal matrix. Let the matrix Λ_D be the diagonal matrix such that $\Lambda_D^2 = L_D$. Then the matrix

$$M = F^T \Lambda_D F - B \tag{17}$$

is the symmetric solution of the equation (16)

Case 4. Let us consider the equation:

$$M^2 + MB^T + BM + A = O \tag{18}$$

After the change $M = N - B$, $M^T = N^T - B^T$, $D = BB^T - A$ we get the equation:

$$NN^T = D \tag{19}$$

The symmetric matrix D is similar to the diagonal matrix L_D . Therefore $D = F^T L_D F$, where the matrix F is the orthogonal matrix. Let the matrix Λ_D be the diagonal matrix such that $\Lambda_D^2 = L_D$. Then we get the solution of the equation (19):

$$N = F^T \Lambda_D G, \tag{20}$$

where the matrix G is any orthogonal matrix. Since M is the symmetric matrix, we obtain the equality: $N - B = N^T - B^T$. After the change (20) we get the equation:

$$F^T \Lambda_D G - G^T \Lambda_D F = B - B^T$$

After the change $X = GF^T$, $P = F(B - B^T)F^T$ finally we get the equation:

$$\Lambda_D X - X^T \Lambda_D = P, \tag{21}$$

where the matrix P is the skew-symmetric matrix. From the equation (21) we want to find the orthogonal complex matrix X . Let us see the matrices $X = (x_{ij})$, $P = (p_{ij})$, $\Lambda_D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$. From the equation (21) we get the system of $\frac{1}{2} n(n - 1)$ linear equations:

$$\lambda_i x_{ij} - \lambda_j x_{ji} = p_{ij}, \quad 1 \leq i < j \leq n \tag{22}$$

After solving the system (22) we get the solution of the equation (18):

$$M = F^T \Lambda_D X F - B \tag{23}$$

Case 5. Let the matrix C of the equation (10) be the symmetric positive-definite matrix, therefore $C = QQ^T$. After the change

$$M_1 = Q^T M Q, \quad B_1 = Q^T B (Q^T)^{-1}, \quad A_1 = Q^T A Q$$

we obtain the equation similar to the equation (18). Let M_1 be the solution of the equation

$$M_1^2 + M_1 B_1^T + B_1 M_1 + A_1 = O$$

Then the matrix

$$M = (Q^T)^{-1} M_1 Q^{-1} \tag{24}$$

is the symmetric solution of the equation (10).

Let us consider the second equation of the system (8). In this equation the matrix U is the Jordan normal form of the matrix $B^T + CM$. Hence this equation always has the solution.

Let us consider the third equation of the system (8). The unknown matrix K is symmetric. Let us consider matrices

$$U = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ \varepsilon_1 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_{n-1} & v_n \end{pmatrix}, \quad \varepsilon_i = 0, \quad 1, \quad i = \overline{1, n-1};$$

$$LCL^T = (c_{ij}), \quad K = (k_{ij}).$$

From the third equation of the system (8) we get the system of $\frac{1}{2} n(n + 1)$ linear equations:

$$(v_i + v_j)k_{ij} + \varepsilon_{i-1}k_{i-1,j} + \varepsilon_{j-1}k_{i,j-1} = c_{ij}, \quad k_{ij} = k_{ji}, \quad \varepsilon_0 = k_{0j} = k_{i0} = 0,$$

$$1 \leq i \leq j \leq n.$$

The determinant of the system is $\prod_{1 \leq i \leq j \leq n} (v_i + v_j)$. If the determinant is nonzero, then the system has the unique solution. We consistently find the unknowns of the system.

Thus we find the solution of the system of matrix equations.

Example. Let

$$A = \begin{pmatrix} 1 & -8 \\ -8 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}, \quad C = E$$

be the matrices of the Hamiltonian (2). Therefore

$$D = B^2 - A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \Lambda_D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Thus the matrices

$$M = \Lambda_D - B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, L = E, U = \Lambda_D, K = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/6 \end{pmatrix}$$

are the solution of the system (8). So

$$S(\mathbf{x}, \mathbf{p}) = 1/2 \mathbf{p}^T K \mathbf{p} + \mathbf{p}^T L \mathbf{x} + 1/2 \mathbf{x}^T M \mathbf{x} = 1/8 p_1^2 + 1/12 p_2^2 + p_1 x_1 + p_2 x_2 + 1/2 x_1^2 + 2 x_1 x_2$$

is the generation function of the canonical transformation. Hence

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^T U \mathbf{q} = 2 p_1 q_1 + 3 p_2 q_2$$

is the new Hamiltonian.

4 Conclusions

We obtain the new method of normalization of the quadratic Hamiltonian. With this method we can investigate the stability of the solution of the Hamiltonian systems.

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