Nonlinear analysis of flexible plates lying on elastic foundation

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Abstract. This article describes numerical procedures for analysis of flexible rectangular plates lying on elastic foundation. Computing models are based on the theory of plates with account of transverse shear deformations. The finite difference energy method of discretization is used for reducing the initial continuum problem to finite dimensional problem. Solution procedures for nonlinear problem are based on Newton-Raphson method. This theory of plates and numerical methods have been used for investigation of nonlinear behavior of flexible plates on elastic foundation with different properties.

1 Introduction

The design and analysis of flexible plates lying on elastic foundation constitute a complex and important domain of structural mechanics. In the analysis of plates on elastic foundation Winkler model with combination of Kirchhoff model is usually used [1-9]. More affective numerical algorithm can be created on the theory of flexible plates with account of transverse shear deformations [10-12]. This model allows to analysis an average thickness plates and plates with a low shear stiffness. In addition, the moments and transverse forces are expressed only in terms of the first derivatives of the displacements while the energy functional of the system contains no derivatives higher than the first order as in the theory of elasticity.

The main aim of this work was to develop an efficient numerical algorithm for the analysis of flexible plates on elastic foundation with account of transverse shear deformations by using a finite difference energy method and the continuation method, and investigation of their strain state.

2 Methods

A rectangular plate is considered in the Cartesian coordinate system (Fig.1). Geometric relationships between the components of strains vector $\boldsymbol{\varepsilon} = (\varepsilon_{11}\varepsilon_{22}\varepsilon_{12}\kappa_{11}\ \kappa_{22}\kappa_{12}\varepsilon_{13}\varepsilon_{23})^T$ and the components of displacements vector $\mathbf{u} = (u\ v\ \ \theta_1\ \theta_2\ w)^T$ have the form:

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$$\varepsilon_{11}=\frac{\partial u}{\partial x}+\frac{1}{2}\theta_1^2;\ \varepsilon_{22}=\frac{\partial v}{\partial y}+\frac{1}{2}\theta_2^2;\ \varepsilon_{12}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\theta_1\theta_2;$$
$$\kappa_{11}=\frac{\partial \theta_1}{\partial x};\ \kappa_{22}=\frac{\partial \theta_2}{\partial y};\ \kappa_{12}=\frac{\partial \theta_2}{\partial x}+\frac{\partial \theta_1}{\partial y};\ \varepsilon_{13}=\frac{\partial w}{\partial x}+\theta_1;\ \varepsilon_{23}=\frac{\partial w}{\partial y}+\theta_2. \quad (1)$$

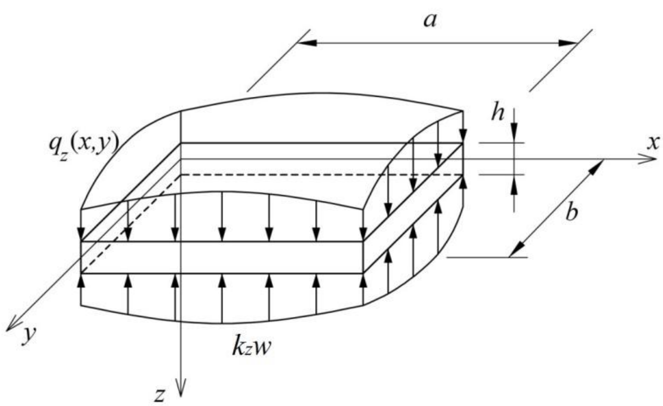

Fig. 1. Plate on elastic foundation

The vector of forces $\mathbf{N}=\left(N_{11}\,N_{22}\,N_{12}\,M_{11}\,M_{22}\,M_{12}\,Q_{13}\,Q_{23}\right)^T$ is associated with the components of strains vector $\boldsymbol{\varepsilon}$ known relation:

$$\mathbf{N}=\mathbf{D}\boldsymbol{\varepsilon}\,, \quad (2)$$

where $\mathbf{D}$ is an elasticity matrix.

Lagrange functional of the theory of flexible plates on elastic foundations based on geometry (1) and physical (2) dependences is constructed. The expression for the functional is:

$$\Pi(\mathbf{u})=\frac{1}{2}\iint_{\Omega}\left(\mathbf{N}^T\boldsymbol{\varepsilon}+k_z w^2\right)d\Omega-\iint_{\Omega}\mathbf{q}^T\mathbf{u}\,d\Omega\,, \quad (3)$$

where k_z is a compression ratio; $\mathbf{q}=\left(q_1\,q_2\,m_1\,m_2\,q_z\right)^T$ is an external load vector whose components correspond to the five components of the displacement vector.

The equilibrium equation (Euler's equation) and the natural boundary conditions follow from the condition of minimum of functional (3). The geometric relations (1) can be written relative to the middle plane and go from shear model to the classical Kirchhoff model if we put $\theta_1=-\frac{\partial w}{\partial x},\ \theta_2=-\frac{\partial w}{\partial y}$. In this case, Euler equation being the equation of equilibrium regarding the function w in linear formulation has the form [13]:

$$\frac{Eh^3}{12\left(1-\nu^2\right)}\nabla^4 w+k_z w=q_z\,, \quad (4)$$

where E – modulus of elasticity of the material of the plate; h – a thickness; ν – Poisson's ratio.

For discretization of the problem we use the finite difference energy method [10, 15, 16]. In this method a grid is applied on the region Ω of the changing of independent variables. The desired function **u**, delivers the stationary value to the functional (3), approximately determined by its values at the nodes, and the derivatives of **u** are replaced by finite differences. The first order derivatives of the grid functions and values of the function at the grid cell are calculated by the formulas:

$$\begin{aligned}
&\frac{\partial u}{\partial x}=\frac{1}{2h_i}\left(u_{i+1\,j+1}+u_{i+1\,j}-u_{i\,j+1}-u_{i\,j}\right);\\
&\frac{\partial u}{\partial y}=\frac{1}{2h_j}\left(u_{i+1\,j+1}+u_{i\,j+1}-u_{i+1\,j}-u_{i\,j}\right);\\
&u_{i+\frac{1}{2}\,j+\frac{1}{2}}=\frac{1}{4}\left(u_{i\,j}+u_{i+1\,j}+u_{i\,j+1}+u_{i+1\,j+1}\right),
\end{aligned}\tag{5}$$

where i, j – node numbers of the grid; h_i, h_j – steps of the grid in the directions of the x and y axes, respectively. Using formulas (5) and replace integration by summation over the area Ω occupied by the plate, the functional (3) reduces to its discrete analog, which is a scalar function of a vector argument.

The solution of the nonlinear problem is performed by the Newton-Raphson method with a step change of the load parameter [14, 15]:

$$\begin{aligned}
&\nabla^2 W(\mathbf{u}_n^m)\Delta\mathbf{u}_{n+1}^m=\mathbf{Q}p^m-\nabla W(\mathbf{u}_n^m);\\
&\mathbf{u}_{n+1}^m=\mathbf{u}_n^m+\tau_n\Delta\mathbf{u}_{n+1}^m,
\end{aligned}\tag{6}$$

where n - an iteration number; m - a number of the step of the load; $\mathbf{u}_n^m$ - the values of the unknown functions on the iteration number n on step m; $\mathbf{u}_{n+1}^m$ - the values of the unknown functions on the iteration number n+1 on step m; $\Delta\mathbf{u}_{n+1}^m$ - the increment functions on iteration number n+1 on step m; p^m – current value of the load parameter; $\nabla^2 W(\mathbf{u}_n^m)$ - the Hessian (matrix of second derivatives) of the discrete analogue of the potential energy of deformation $W(\mathbf{u}_n^m)$; $\nabla W(\mathbf{u}_n^m)$ - the gradient (vector of residual) of the function $W(\mathbf{u}_n^m)$; τ_n - iteration parameter governing the step size.

The components of the gradient vector and the Hessian matrix elements are calculated using the formulas [10, 14, 15] based on expressions (1) and (2):

$$\begin{aligned}
&\frac{\partial W(\mathbf{u})}{\partial u_i}=\iint_\Omega \boldsymbol{\varepsilon}^T\mathbf{D}\frac{\partial\boldsymbol{\varepsilon}}{\partial u_i}d\Omega;\\
&\frac{\partial^2 W(\mathbf{u})}{\partial u_i\partial u_j}=\iint_\Omega\left(\left(\frac{\partial\boldsymbol{\varepsilon}}{\partial u_i}\right)^T\mathbf{D}\frac{\partial\boldsymbol{\varepsilon}}{\partial u_j}+\boldsymbol{\varepsilon}^T\mathbf{D}\frac{\partial^2\boldsymbol{\varepsilon}}{\partial u_i\partial u_j}\right)d\Omega,
\end{aligned}\tag{7}$$

where **u** and **ε** are respectively the grid vector functions of the nodal displacements and deformations.

In formulas (7) the first and second derivatives of the vector $\boldsymbol{\varepsilon}$ with respect to the nodal displacements are determined by substituting the unit vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ in the linear $\Delta\boldsymbol{\varepsilon}^L$ and nonlinear part $\Delta\boldsymbol{\varepsilon}^{NL}$ of the increment of deformation:

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial u_i} = \Delta\boldsymbol{\varepsilon}^L(\mathbf{e}_i);$$

$$\frac{\partial^2 \boldsymbol{\varepsilon}}{\partial u_i \partial u_j} = \Delta\boldsymbol{\varepsilon}^{NL}(\mathbf{e}_i + \mathbf{e}_j) - \Delta\boldsymbol{\varepsilon}^{NL}(\mathbf{e}_i) - \Delta\boldsymbol{\varepsilon}^{NL}(\mathbf{e}_j). \quad (8)$$

The differential formulas approximations for calculations of the first and second derivatives of $W(\mathbf{u})$ were also used in numerical analysis:

$$\frac{\partial W(\mathbf{u})}{\partial u_i} = \frac{1}{2\delta} \sum_{t=1,-1} t W(\mathbf{u} + t\delta \mathbf{e}_i);$$

$$\frac{\partial^2 W(\mathbf{u})}{\partial u_i \partial u_j} = \frac{1}{4\delta^2} \sum_{t_1=1,-1} t_1 \sum_{t_2=1,-1} t_2 W(\mathbf{u} + t_1\delta\mathbf{e}_i + t_2\delta\mathbf{e}_j), \quad (9)$$

where $\delta = (10^{-3} - 10^{-5})\|\mathbf{u}\|$ - some small deviation; $\|\mathbf{u}\| = \sqrt{\mathbf{u}^T\mathbf{u}}$ - spherical norm of the vector of displacements and $\mathbf{e}_i$, $\mathbf{e}_j$ - unit vectors.

Formulae (9) permit organization of a flexible computational process which can be easily reorganized depending on the specific type of variational functional. However, they lead to computation of the function $W(\mathbf{u})$ several times, which involves considerable computer time. A more economic and accurate approach is that based on computation of the first and second derivatives of $W(\mathbf{u})$ in terms of their own obvious expressions (7), (8).

For solving the system of linear algebraic equations of the finite difference energy method LDL^T – factorization is applied. The compact form of storage tapes matrix by columns in one-dimensional array is used.

3 Results

Consider the solution of several problems about the behavior of a flexible plate on elastic foundation at various values of the compression ratio (Fig.1). A plate is considered in dimensionless coordinates $\xi = x/a$, $\eta = y/b$. Dimensionless variables and parameters are the following: $W = w/h$; $k = \frac{a^4(1-\nu^2)}{Eh^3}k_z$; $q = \frac{a^4(1-\nu^2)}{Eh^4}q_z$; $N_\xi = \frac{a^2(1-\nu^2)}{Eh^3}N_x$; $M_\xi = \frac{12a^2(1-\nu^2)}{Eh^4}M_x$; $Q_\xi = \frac{2a(1+\nu)}{Eh^2}Q_x$; $\lambda = a/b$, where a and b – dimensions of a plate in a plan; h – a thickness of the plate; E – elasticity modulus of the material of the plate; ν – Poisson's ratio; k – a compression ratio; q_z – an intensity of the transverse distributed load; N_x – normal force; M_x – bending moment; Q_x – shear force.

The following baseline data is taken in the calculations: $k = 0{,}443 \cdot 10^{-5} k_z$; $q = 0{,}443 \cdot 10^{-3} q_z$; $N_\xi = 0{,}433 \cdot 10^{-5} N_x$; $M_\xi = 5{,}2 \cdot 10^{-3} M_x$; $Q_\xi = 1{,}24 \cdot 10^{-7} Q_x$;

$\lambda = 1$. Boundary conditions correspond to hinged bearing at the contour.

For the solution of nonlinear problems step-by-step loading and the Newton-Raphson procedure (6)-(9) for each step are used. The value of load increment at each step was taken equal to Δq=4.33.

Equilibrium curves of the flexible plate (*w*/*h* – deflection in the center) lying on elastic foundations with different compression ratios which conform to the coefficients of a real soils (sand, clay etc.) are shown in Fig. 2. Values of deflections, bending moments and normal forces in the center of the plate for $q / \Delta q = 50$ are given in Table 1.

Table 1. Deflections, bending moments and normal forces in the center of the plate

k	W	M_ξ	N_ξ
0	2.658	20.3	8.31
43.3	2.278	17.4	6.19
86.6	1.934	14.6	4.50
129.9	1.635	11.9	3.22
173.2	1.387	9.52	2.35
259.8	1.150	7.38	1.68

Analysis of the results shows that with increasing values of the parameter *k*, the deformation of the plate takes on a linear character, despite the fact that the deflections reach values comparable with the thickness (Fig. 2, curve 6).

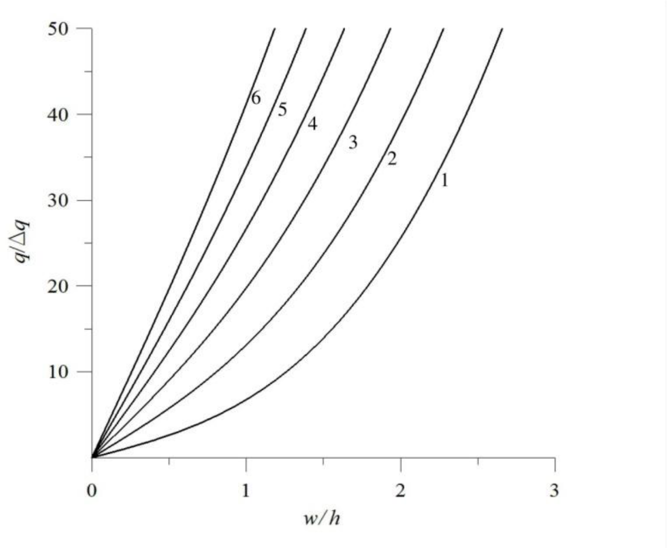

Fig. 2. Equilibrium curves of the flexible plate on elastic foundations with different compression ratios: 1 – *k*=0; 2 – *k*=43.3; 3 – *k*=86.6; 4 – *k*=129.9; 5 – *k*=173.2; 6 – *k*=259.8

In Fig. 3 diagrams showing the change of deflections at the center of the plate with *k*=259.8 are presented. Diagram 1 shows a non-linear solution, diagram 2 – linear solution.

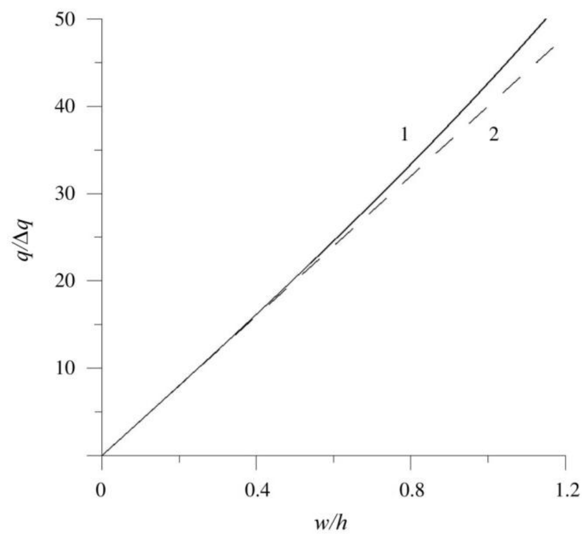

Fig. 3. The change of deflections at the center of the plate: 1 – nonlinear solution; 2 – linear solution

In Table 2 the deflections and in Table 3 bending moments and shear forces in the plate along the line $y = b/2$ are shown.

Table 2. Deflections in the plate

Coordinate ξ	Deflection W	
	Linear solution	Nonlinear solution
0	0	0
1/8	0.5954	0.5590
1/4	1.002	0.9323
3/8	1.197	1.105
1/2	1.250	1.150

Table 3. Bending moments and shear forces in the plate

Coordinate ξ	Linear solution		Nonlinear solution	
	M_ξ	Q_ξ	M_ξ	Q_ξ
1/16	8.79	$38.8 \cdot 10^{-4}$	8.74	$37.8 \cdot 10^{-4}$
3/16	16.7	$0.630 \cdot 10^{-4}$	16.2	$-1.55 \cdot 10^{-4}$
5/16	12.9	$-9.25 \cdot 10^{-4}$	11.7	$-9.96 \cdot 10^{-4}$
7/16	8.68	$-4.49 \cdot 10^{-4}$	7.38	$-4.36 \cdot 10^{-4}$

4 Discussion

The value of the deflections obtained according to a linear solution is greater than the corresponding values obtained with account of geometric nonlinearity. The maximum relative difference is 8,7% for the central point. A similar pattern holds for bending moments where the maximum relative difference is 17,6%. This quasilinear behavior of the flexible plate lying on elastic foundation can be due to the presence of the term *kw* included in the equation of equilibrium and created a certain kind of shell effect. Epures of bending moments and shear forces based on the data of Table 3, have a view typical for shallow shells on a rectangular plan with an action of a lateral uniformly distributed load.

Conclusions

For the calculation scheme considered above and parameters characterizing elastic foundation, it is possible to use a geometrically linear theory in those cases where the deflections of the plate are not small values and comparable with a plate thickness.

The presented numerical method for solving of nonlinear problems of the theory of flexible plates on elastic foundation is sufficiently universal and has a good convergence. The developed technical theory of plates is well suited not only for an analysis of thin-walled structures made of composite anisotropic materials (where the effect of shear changes the results both quantitatively and qualitatively), but also for an analysis of structures made of isotropic materials mainly with a view to simplifying the algorithms for numerical calculations. The presented algorithm can be used in particular in the analysis of the interaction of metal tanks with the ground.

References

1. E. Winkler, *Die Lehre von der Elasticitaet und Festigkeit* (Dominicus, Prag,1867)
2. M.I. Gorbunov-Posadov, T.A. Malikova, V.I. Solomin, *Analysis of Structures on Elastic Foundation* (Stroyizdat, Moscow, 1984)
3. G.S. Vardanyan, V.I. Andreev, N.M. Atarov, A.A. Gorshkov, *Soprotivlenie Materialov s Osnovami Teorii Uprugosti i Plastichnosti* (Infra-M, Moscow, 2014)
4. V.I. Andreev, E.V. Barmenkova, A.V. Matveeva, Vestnik MGSU **12**, 31 (2014)
5. E.S. Egorova, A.V. Ioskevich, V.V. Ioskevich, K.N. Agishev, V.Yu. Kozhevnikov, Construction of Unique Buildings and Structures **3**, 31 (2016)
6. N.S.V. Kameswara Rao, *Foundation Design: Theory and Practice* (John Wiley & Sons, Singapore, 2011)
7. E. Ventsel, T. Krauthammer, *Thin Plates and Shells* (Marcel Dekker, New York, 2001)
8. J.N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, (CRC Press, New York, 2007)
9. J.R. Xiao, Comput. Mech., **27**, 1 (2001)
10. I.E. Mileikowskii, S.I. Trushin, *Analysis of Thin-Walled Structures* (A.A. Balkema Publishers, Rotterdam, 1994**)**.
11. S.I. Trushin, *Structural Mechanics: Finite Element Method* (INFRA-M, Moscow, 2016)
12. S. Trushin, D. Goryachkin, Procedia Engineering **153**, 781 (2016)
13. S. P. Timoshenko, S. Woinowsky-Krieger, *Plates and Shells* (Librokom, Moscow, 2009)
14. A.S. Ivanov, S.I. Trushin, Proceedings of the SEIKEN-IASS Symposium, 89 (1993)
15. S.I. Trushin, Structural Mechanics and Analysis of Constructions **4**, 27 (2007)
16. S.I. Trushin, S.I. Zhavoronok, Space Structures 5, 1527 (2002)