The Mean-CVaR Model for Portfolio Optimization Using a Multi-Objective Approach and the Kalai-Smorodinsky Solution

R. Aboulaich[1], *R.* Ellaia[1], *S.* Elmoumen[2], *A.* Habbal[3], and *N.* Moussaid[4],[*]

[1]*LERMA, Mohammadia School of Engineering, Mohammed V University Rabat, Morocco*

[2]*LIMSAD, Mathematics and computing Department, Ain Chock Sciences Faculty, Casablanca, Morocco*

[3]*INRIA, University of Nice Sophia-Antipolis, France*

[4]*LMA, Hassan II University Casablanca, FST, Mohammedia, Morocco*

Abstract. The purpose of this work is to present a model for portfolio multi-optimization, in which distributions are compared on the basis of tow statistics: the expected value and the Conditional Value-at-Risk (CVaR), to solve such a problem many authors have developed several algorithms, in this work we propose to find the efficient boundary by using the Normal Boundary Intersection approach (NBI) based on our proposed hybrid method SASP, since the considered problem is multi-objective, then we find the Kalai-smorodinsky solution.

1 Introduction

Mean-risk models are still the most widely used approach in the practice of portfolio selection. With mean-risk models, return distributions are characterized and compared using two statistics: the expected value and the value of a risk measure. Thus, mean-risk models have a ready interpretation of results and in most cases are convenient from a computational point of view. On the other hand in the practice of describing a distribution by just two parameters involves great loss of information.

It is well known that the risk measure plays an important role in making the decisions. Variance was the first risk measure used in mean-risk models (Markowitz 1952) and, in many proposals of new risk measures (see for example Fishburn (1977), Yitzhaki (1982), Konno and Yamazaki (1991), Ogryczak and Ruszczynski (1999, 2001), Rockafellar and Uryasev (2000, 2002)). Another risk measure, the Conditional Value-at-Risk (CVaR))[16],[20], is growing in popularity. CVaR has attractive theoretical properties: it controls the magnitude of losses beyond Value-at-Risk (VaR) and it is coherent (see for example Artzner 1999, Acerbi and Tasche 2002, Tasche 2002, Pflug 2000, Rockafellar and Uryasev 2002). In this paper we propose to solve the problem of portfolio selection, which is a multi-objective problem, first by using the NBI approach [8] based on SASP method [7], implemented in Matlab to find the efficient boundary, after we use the game theory [3, 4][9][2][1] to determine the Kalai-smorodinsky solution.

2 The portfolio selection problem

The problem of portfolio selection with one investment period is an example of the general problem of deciding between random variables when larger outcomes are preferred. Decisions are required on the amount (proportion) of capital to be invested in each of a number of available assets such that at the end of the investment period the return is as high as possible. Consider a set of n assets, with asset j in $\{1, ..., n\}$ giving a return R_j at the end of the investment period. R_j is a random variable, since the future price of the asset is not known. Let x_j be the proportion of capital invested in asset j ($x_j = w_j/w$ where w_j is the capital invested in asset j and w is the total amount of capital to be invested), and let $x = (x_1, ..., x_n)$ represent the portfolio resulting from this choice. This portfolio's return is the random variable: $R_x = x_1 R_1 + ... + x_n R_n$, with distribution function $F(r) = P(R_x \leq r)$ that depends on the choice $x = (x_1, ..., x_n)$.

To represent a portfolio, the weights $(x_1, ..., x_n)$ must satisfy a set of constraints that forms a feasible set $\mathbf{A}$ of decision vectors. The simplest way to define a feasible set is by the requirement that the weights must sum to 1 and short selling is not allowed. For this basic version of the problem, the set of feasible decision vectors is:

$$\mathbf{A} = \{(x_1, ..., x_n) / \sum_{i=1}^{n} x_j = 1, x_j \geq 0, \forall j \in \{1, ..., n\}\}$$

If we consider a different portfolio defined by the decision vector $y = (y_1, ..., y_n) \in \mathbf{A}$, where y_j is the proportion of capital invested in asset j. The return of this portfolio is given by the random variable $R_y = y_1 R_1 + ... + y_n R_n$.

The problem of choosing between portfolio $x = (x_1, ..., x_n)$ and portfolio $y = (y_1, ..., y_n)$ becomes the problem of

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[*]e-mail: moussaid_noureddine@yahoo.fr

choosing between random variables R_x and R_y. The criteria by which one random variable is considered "better" than another random variable need to be specified and models for choosing between random variables (models for preference) are required. a lot of models are proposed, the markowitz model, the mean model and the mean variance model.

3 Conditional Value-at-Risk

Let R_x be a random variable representing the return of a portfolio x over a given holding period and $\alpha \in (0, 1)$ a percentage which represents a sample of "worst cases" for the outcomes of R_x.

The definition of CVaR at the specified level α is the mathematical transcription of the concept "average of losses in the worst α of cases" (Acerbi and Tasche 2002), where a "loss" is a negative outcome of R_x (thus the loss associated with R_x is described by the random variable $-R_x$). An important result, proved by Rockafellar and Uryasev (2000, 2002), and independently by Ogryczak and Ruszczynski (2002), is that the CVaR of a random variable R_x can be calculated by solving an optimization problem. Moreover, CVaR can be minimized over the set of feasible decision vectors. These results are summarized below:

Let R_x be a random variable depending on a decision vector x that belongs to a feasible set $\mathbf{A}$, and $\alpha \in (0, 1)$. Consider the function:

$$F_\alpha(x, \upsilon) = \frac{1}{\alpha} E\{[-R_x + \upsilon]^+\} - \upsilon$$

Then:
(a) As a function of υ, F_α is finite and continuous (hence convex) and

$$CVaR_\alpha(R_x) = \min_{\upsilon \in R} F_\alpha(x, \upsilon).$$

(b) Minimizing $CVaR_\alpha$ with respect to $x \in A$ is equivalent to minimizing F_α with respect to $(x, \upsilon) \in A \times R$:

$$\min_{x \in A} CVaR_\alpha(R_x) = \min_{(x,\upsilon) \in A \times R} F_\alpha(x, \upsilon). \quad (1)$$

(c) $CVaR_\alpha(R_x)$ is convex with respect to x and $F_\alpha(x, \upsilon)$ is convex with respect to (x, υ).

The Mean-CVaR model

Let E be the expected value of R_x, the Mean-CVaR model can be formulated for the portfolio selection problem as follows:

$$\min_{x \in A} F = (f_1, f_2) \quad (2)$$

Where $f_1 = CVaR$ and $f_2 = -E$
Since 1, the problem 2 becomes:

$$\min_{x \in A} (F_\alpha(x, \upsilon), -E) \quad (3)$$

4 Normal Boundary Intersection

Let x_i^* and f_i^* denote respectively the minimizer and minimum value of the f_i and let F^* denote the shadow minimum, i.e., the vector whose components are f_i^*. Consider the shifted pay-off matrix Φ whose i^th column is $F(x_i^*) - F^*$. The Convex Hull of Individual Minima or CHIM is defined as the set of points that are convex combinations of the columns of Φ , i.e., $\{\Phi\beta : \beta_i \geq 0, \sum_i \beta_i = 1\}$.

For a two dimensional problem illustrated in Figure 1, CHIM is represented by segment AB.
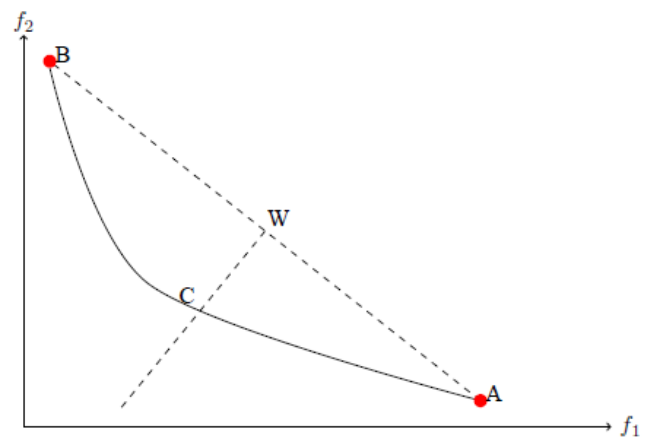

Figure 1. An illustrative integrated design

The idea behind NBI is to pick an even spread of points on the CHIM (for example W in Fig.1), and find the intersection point between the efficient frontier and a set of parallel normals emanating from the chosen set of points on the CHIM (C in Fig.1). Given a convex combination parameter vector β, and a normal direction n pointing towards the origin, the point of intersection between the normal emanating from $\Phi\beta$ and the efficient frontier can be found by solving the following NBI_β subproblem :

$$\begin{cases} \min_{x \in A} t \,; \\ \Phi\beta + tn = F(x) - F^*. \end{cases} \quad (4)$$

By solving subproblem NBI_β 4 for different settings of β, various points on the efficient frontier can be generated. The advantage of the β parameter is that an even spread of β parameters corresponds to an even sped of points on the CHIM.

5 Kalai Smorodinsky solution

In this paper, we research the Kalai Smorodinsky solution [18], witch is the intersection between the Pareto front and the line passing through the utopian point Ut and the disagreement point D. Note,

$$Ut = \begin{pmatrix} f_1(x^*) \\ f_2(y^*) \end{pmatrix}, D = \begin{pmatrix} f_1(y^*) \\ f_2(x^*) \end{pmatrix} \text{ and } \tau = \frac{Ut - D}{\| Ut - D \|}$$

where,

$$\begin{cases} x^* = Arg\min_x f_1(x) \\ y^* = Arg\min_y f_2(y). \end{cases} \quad (5)$$

Theorem 1 *A solution is symmetric, Pareto optimal[9][2][1], invariant under affine transformations of utility scale, and individually monotonic if and only if it is the Kalai-Smorodinsky solution [18].*

To search the Kalai Smorodinsky solution [5], we proposed to solve the following problem:

$$(KS)\begin{cases} \max_{y,t} t, \\ s.c\ F(x) = D + t\tau, \end{cases} \quad (6)$$

To solve such problem we propose to use the SASP method [7].

6 Numerical Example

6.1 An application to portfolio optimization

Let (SAMIR, ATWB, AGM, LAM, GAZ, IAM, ONA) be the assets, quoted in the Casablanca Stock Exchange, with history of 5 years (see the following Figure).
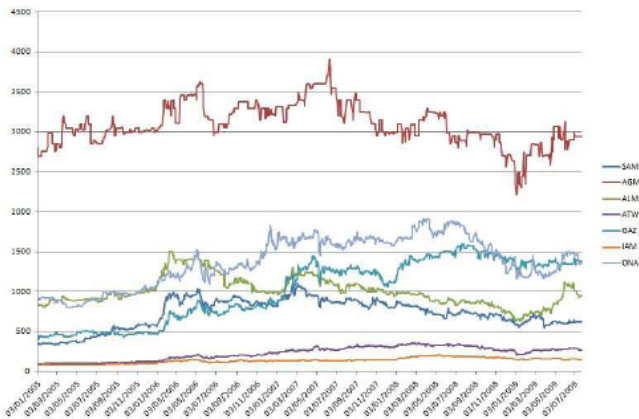

Figure 2. An illustrative integrated design

First we are going to solve the problems (8) and (9):

$$\min_{x \in A} CVaR \quad (7)$$

and

$$\min x \in A - E \quad (8)$$

Then we have to find the efficient frontier with the NBI method based on the SASP method, for this, we have to solve the problems below for deferent β with $\beta \in \{\beta \in IR^n, \sum_{i=1}^{n} \beta_i = 1, \beta_i \geq 0\}$:

$$\begin{aligned} \max \quad & t \\ s.t. \quad & \Phi\beta + tn = F(x) - F^* \\ & x \in A \end{aligned}$$

The Kalai Smorodinsky solution can be find by solving the following problem:

$$(KS)\begin{cases} \max_{x,t} t, \\ s.c\ F(x) = D + t\tau, \end{cases} \quad (9)$$

6.2 Results

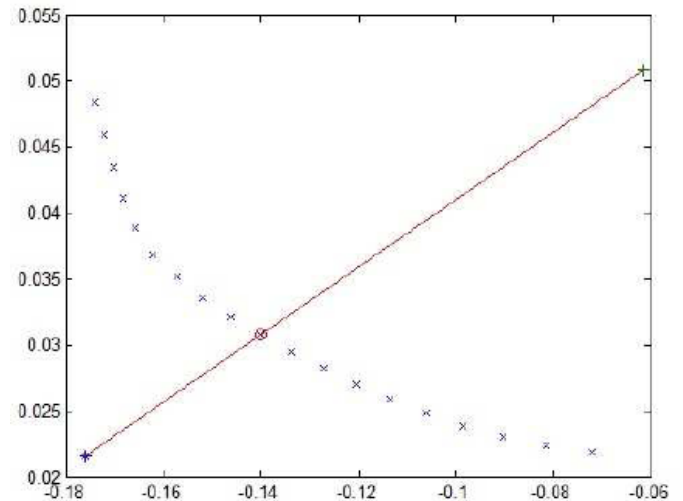The results are provided by the following figure:

Figure 3. An illustrative integrated design

As we can see clearly the NBI parametrization of the efficient front naturally bears the property of even spread, and the Kalai Smorodinsky solution is closest to the ideal solution (maximum E, minimum CVaR), the Kalai Smorodinsky portfolio "PF" (36% in ATW, 37% in GAZ and 27% in IAM) history is given with the other histories in the following figure:
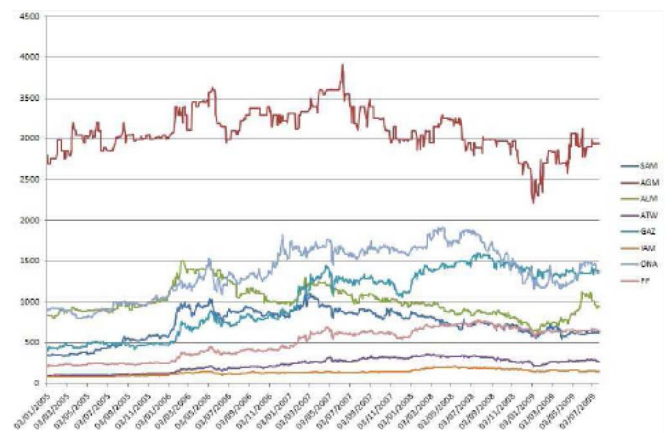

Figure 4. An illustrative integrated design

If we choose another optimal portfolio, for example the portfolio corresponding to (0,1061, 0,0248) we ob-

tain (13,81% in AGM,5,02% ALM, 22,93% ATW, 25,5% GAZ and 32,75 IAM) and for (0,1722 , 0,0459) we obtain (14,21% ATW, 85,79% GAZ), we can't have an equilibrium between maximum E and minimum CVaR.

7 Conclusion

In this work, we proposed a solution of bi-objective model in order to obtain the optimal portfolio. First we used the Normal Boundary Intersection approach (NBI) to compute the efficient front, and then the optimal solution is given by Kalai Smorodinsky solution. The obtained results show the efficiency of the proposed approach.

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