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Modelling guided waves in anisotropic plates using the Legendre polynomial method

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Abstract. A numerical method to compute phase dispersion curve in unidirectional laminate is described. The basic feature of the proposed method is the expansion of fields quantities in single layer on different polynomial bases. The Legendre polynomial method avoid to solve the transcendental dispersion equation of guided wave. Guided waves that have very close propagation constants are calculated with great accuracy. Numerical solution of dispersion relation are calculated for guided waves propagation in orthotropic unidirectional fiber composites. The validation of the polynomial approach is depicted by a comparison between the associated solution and those obtained using Transfer matrix method.

1 Introduction

Composite materials play an important role in many structural components, such as auto parts, boat hulls, and aircraft structures [1].In recent years, much attention has been paid to nondestructive testing method for composite. Ultrasonic guided waves is effective way to detect the presence of structure defect such as damage, disbands and delaminations. For any analysis of guided waves propagation in structures, the velocity of the guided wave mode are essential for further study [2]. In many cases, accurate calculation of wave's speeds are very benefit to conduct experimental studies.

It is also true that the multi-mode characters, dispersion of guided waves, and the anisotropic behaviour of composite materials, guided waves are exceedingly complex and not easy to analyse. There are many methods to investigate the propagation characteristics of guided waves in composite laminates. These method include the Transfer matrix method (TMM)[3, 4], the Global matrix method (GMM)[5], the scattering matrix method(SMM)[6], the stiffness transfer matrix method(STMM)[1], reverberation-ray matrix method(RMMM)[7, 8], the finite element method (FEM)[9-11], and the semi-analysis finite element method (SAFE) [12]. Guided waves propagation can then be found by imposing appropriate boundary. However, the search of root still remains a rather difficult task, and part of them may be missed.

In order to avoid these drawbacks, an expansion polynomial method has been presented to obtain wave propagation without solving any transcendental equation. Lefebvre et al. [13] studied wave propagation in piezoelectric plates using the Legendre polynomial method since then this approach is used extensively to investigate the guided waves propagation in various structure [14-18]. The key aspect of Legendre polynomial method is the expansion of the unknown displacement quantities on a

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set of functions. The differential wave equation turns into an algebraic eigenvalue problem. The eigenvalues and corresponding eigenvectors are used to acquire the propagation constants by linear combinations of the expansion functions.

In this paper, our work are focused on the derivation and analysis of guided waves equation for unidirectional laminate using the Legendre polynomial. Using the proposed meth, the dispersion curves of different propagation angles with respect to the principal fibre orientation are calculated, and the result solution are compared with the result obtained from Transfer matrix method.

2 Formulation of the problem

Consider the general case of a layer structure of infinite extent in the the x_1 and x_3 directions and thickness H in the x_2 direction, as given in Fig. 1.Fig.1 demonstrate the coordinate system for orthotropic layer, where $o-x_1x_2x_3$ and $o-x_1'x_2'x_3'$ represent global and local coordinate system respectively. For unidirectional fiber reinforced composite laminates, the stiffness matrix can be transformed from the local reference coordinates to the global coordinates by[17] :

$$c_{mnop} = \beta_{mi}\beta_{nj}\beta_{ok}\beta_{pj}c'_{ijkl} \tag{1}$$

Where β_{ij} indicates the cosine of the intersection angle ϕ between the x_i' axis and x_j' axis (i,j=1,2,3).

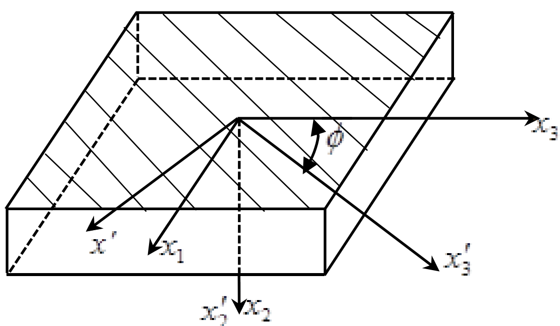


Figure 1. The spatial coordinate system for an orthotropic lamina

The wave motion without considering body force, can be written in the following form:

$$\begin{cases} \nabla\cdot(c:\nabla_s u)=\rho\dfrac{\partial^2 u}{\partial t^2} \\ \nabla_s u=\sigma \end{cases} \tag{2}$$

Where ρ stand for the density, σ stand for the strain tensor, u stand for the displacement vector. Eq.(2) contain 9 equation and 9 unknowns. The 9 unknowns are three displacement and six stress components. However, for simplicity, this unknown variables are expressed by the vector form

$$u=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}^T \mathrm{e}^{-j[kx_1-\omega t]} \tag{3}$$

$$\tau_i=\begin{bmatrix}\sigma_{i1} & \sigma_{i2} & \sigma_{i3}\end{bmatrix}^T \mathrm{e}^{-j[kx_1-\omega t]} \tag{4}$$

Substituting Eq. (3-4) into Eq. (2), the harmonic wave term $\mathrm{e}^{-j[kx_1-\omega t]}$ is omitted, and the governing equations can be rewritten in term of

$$\frac{\partial\tau_2}{\partial x_2}=-\rho\omega^2[I]u-\frac{\partial\tau_1}{\partial x_1}-\frac{\partial\tau_3}{\partial x_3} \tag{5}$$

Eq.(2) can be rewritten in the form

$$\tau_1=\frac{\partial}{\partial x_1}[D_{11}]u+\frac{\partial}{\partial x_2}[D_{12}]u+\frac{\partial}{\partial x_3}[D_{13}]u \tag{6}$$

$$\tau_2 = \frac{\partial}{\partial x_1}[D_{21}]u + \frac{\partial}{\partial x_2}[D_{22}]u + \frac{\partial}{\partial x_3}[D_{23}]u \tag{7}$$

$$\tau_3 = \frac{\partial}{\partial x_1}[D_{31}]u + \frac{\partial}{\partial x_2}[D_{32}]u + \frac{\partial}{\partial x_3}[D_{33}]u \tag{8}$$

Using the Eq.(7), The first order derivatives of the displacement versus x_2 can be expressed as

$$\frac{\partial}{\partial x_2}u = [D_{22}]^{-1}\left(\tau_2 - \frac{\partial}{\partial x_1}[D_{21}]u - \frac{\partial}{\partial x_3}[D_{23}]u\right) \tag{9}$$

At this stage,it is possible to in introduce the state vector $\eta = [u_1 \quad u_2 \quad u_3 \quad \sigma_{21} \quad \sigma_{22} \quad \sigma_{23}]$,the general solution for the state vector can be expressed in the form $\eta = \eta \mathrm{e}^{j(\omega t - kx_1)}$,governing differential equations for the state vector in term of the displacement and stress quantities can be given

$$\frac{\partial \eta}{\partial x_2} = \overline{A}\eta \tag{10}$$

Where

$$\overline{A} = \begin{bmatrix} [D_{22}]^{-1} & jk[D_{22}]^{-1}[D_{21}] \\ jk[D_{12}][D_{22}]^{-1} & -\rho\omega^2[I] - k^2\left[[D_{12}][D_{22}]^{-1}[D_{21}] - [D_{11}]\right] \end{bmatrix}$$

The relation between τ_2 and u can be obtain from Eq.(7)

$$\tau_2 = [D_{22}]\frac{\partial u}{\partial x_2} - jk[D_{21}]u \tag{11}$$

In order to remove the stress state vector from Eq.(10), Substituting Eq.(11) into the Eq.(10) one obtains

$$k^2[D_{11}]u + jk[D_{12}]\frac{\partial u}{\partial x_2} - \rho\omega^2[I]u + [D_{22}]\frac{\partial^2 u}{\partial {x_2}^2} = 0 \tag{12}$$

Considering the flat plate structure as a case, was shown in fig.1. A set of Legendre polynomials are used to represent the unknown displacement in layer. The displacement vector u can be expressed as

$$u = \sum_{n=0}^{N-1}\psi_n P_n(\chi) \tag{13}$$

Where $P_n(\chi)$ is Legendre polynomial, ψ_n is unknown expansion coefficients, and $\chi \in [-1,1]$ being a normalized variable:

$$\chi = \ell(x_2 - {x_2}^0), \quad \ell = 2/h \tag{14}$$

Where,h and ${x_2}^0$ are the thickness of plate and the coordinate of the plate middle point. Using the Eq.(14),The first and second derivatives of u can be written as

$$\frac{\partial u}{\partial x_2} = \ell\frac{\partial u}{\partial \chi} = \ell\sum_{n=0}^{N-2} p_n P_n(\chi) \tag{15}$$

$$\frac{\partial^2 u}{\partial {x_2}^2} = \ell^2\frac{\partial^2 u}{\partial \chi^2} = \ell^2\sum_{n=0}^{N-3} q_n P_n(\chi) \tag{16}$$

with p_n and q_n are expressed by

$$p_m = (2m+1) \sum_{n=m+1,m+3,\cdots}^{N-1} \psi_n \tag{17}$$

$$q_m = \left(\frac{2m+1}{2}\right) \sum_{n=m+2,\ m+4,\cdots}^{N-1} \left(\mathrm{n}(n+1) - m(m+1)\right)\psi_n \tag{18}$$

The introduction of representation Eq.(13) in Eq.(12).Multiplying by Legendre polynomial of order m both sides of this equation, and integrating over χ from -1 and 1,then using the orthogonal property of Legendre polynomial,gives

$$k^2\left[D_{11}\frac{2}{2m+1}\right]\psi_m + jk\ell\left[D_{12}\frac{2}{2m+1}\right]p_m + \left[-\rho\omega^2\frac{2}{2m+1}I\right]\psi_m + \ell^2\left[D_{22}\frac{2}{2m+1}\right]q_m = 0 \tag{19}$$

Eqs.(19) are quadratic in k ,therefor the total number of unknown is $6N$,however we can obtained $3N+3(N-2)$ equation from Eqs.(19),this indicated that the remaining 6 equations are provide by boundary condition.The boundary conditions for the orthotropic plate generally require that: the stress should be zero at the upper and bottom surfaces. The following boundary conditions can be yield

$$\text{at } x_2 = 0: \sum_{n=0}^{N}\left(D_{22}(-1)^{n+1}\ell\frac{n(n+1)}{2} - j(-1)^n k\left[D_{21}\right]\right)\psi_n = 0 \tag{20}$$

$$\text{at } x_2 = h: \sum_{n=0}^{N}\left(D_{22}\ell\frac{n(n+1)}{2} - jk\left[D_{21}\right]\right)\psi_n = 0 \tag{21}$$

The value of Legendre polynomial and their first derivatives at the end of the interval $[-1,1]$ has to be taken into account for deriving the Eqs.(20-21)：

$$P_n(\pm 1) = (\pm 1)^n \tag{22}$$

$$\frac{dP_n(\pm 1)}{dr} = (\pm 1)^{n+1}\frac{n(n+1)}{2} \tag{23}$$

We collected all the relevant equation from Eqs.(19-21),and arranged the unkowns in the column vector $\Omega = [\psi_0, \cdots, \psi_{N-1}]^T$,Where $\psi_k = [u_{1k}, u_{2k}, u_{3k}]^T$,a generalized eigenvalue problem is obtained：

$$k^2\bar{R}\Omega + jk\bar{S}\Omega + \bar{T}\Omega = 0 \tag{24}$$

Here, $\Lambda = jk\bar{R}\Omega$ are introduced to obtained a linear eigenvalue problem from Eq.(24) . The corresponding equations are as follow:

$$\left(\begin{bmatrix}\bar{S} & -I \\ \bar{R} & 0\end{bmatrix} - \frac{j}{k}\begin{bmatrix}\bar{T} & 0 \\ 0 & I\end{bmatrix}\right)\begin{bmatrix}\Omega \\ \Lambda\end{bmatrix} = 0 \tag{25}$$

Where I is the identity matrix. The eigensolutions of k and Ω represent the wavenumber and the displacement vector associated with the corresponding guided wave mode.

3 Numerical examples

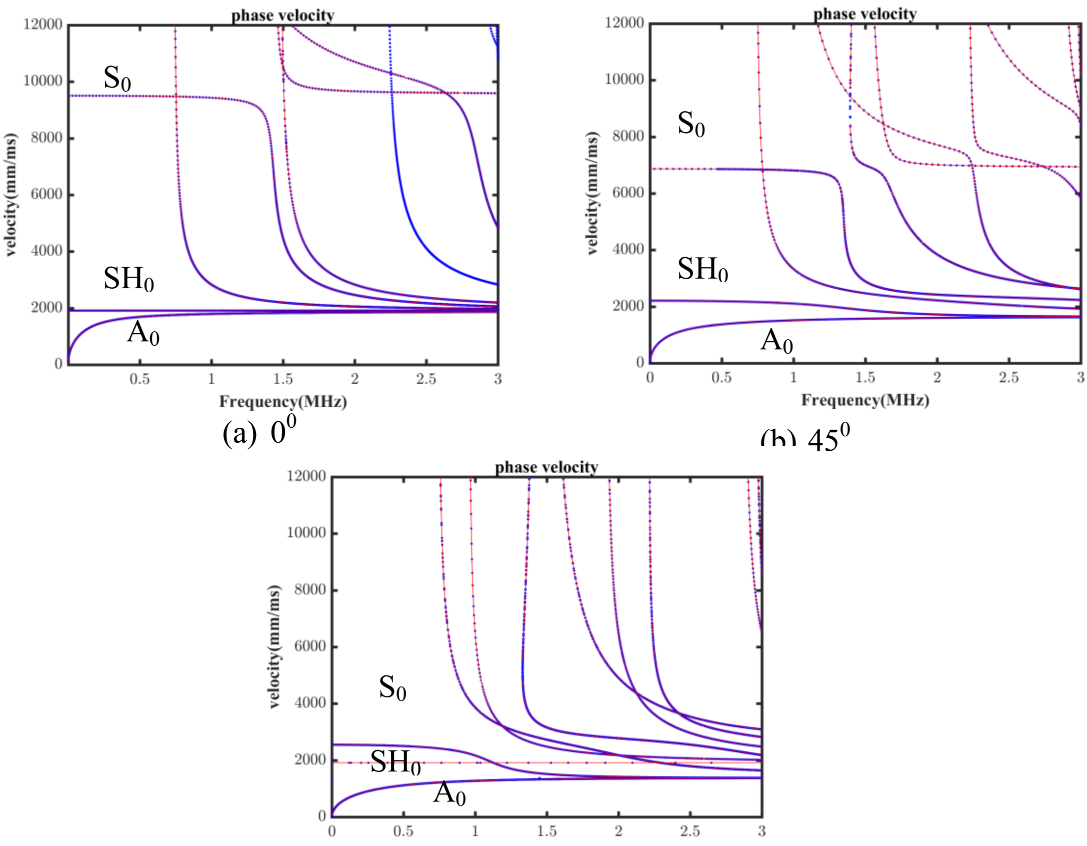
In this section, results are shown for hase velocity dispersion curves, the material used as case study is T300/914 .These material properties are shown in table 1.The lamina thickness is 1 mm.the material density is $\rho = 1560 kg/m^3$. Fig.2 presents the phase velocity results for waves propagating along three different direction of 0^0 ,45 and 90 with respect to the fiber principal direction. The symmetric and anti-symmetric lamb wave are number as 'Sn' and 'An'(n=0,1,2…),respectively. The SH waves are number as 'SHn'.Figure 2 compares the results obtained from LPM with those obtained by TMM for the case of lamb and SH modes in a single plate. As can be observe in Fig.2 that all the modes were found and the LPM's solution are coincident with those obtained by the TMM's solution.

Table1 Elastic constants of T300/914

Unit: C_{ij} (10^9 N/m^2)

C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
143.8	6.2	6.2	13.3	6.5	13.3	3.6	5.7	5.7

From fig 2,it can be observe that lamb and SH wave are decoupled for the principal direction ($\phi = 0^0$ and $\phi = 90^0$) and SH0 is almost non-dispersive in the entire frequency range . For the propagation direction at 45 with respect to the direction x_1 axis, the lamb and SH wave are coupled and the SH0 mode has some degree of dispersion.it can also be found that the velocity of the lamb waves tend to abate with increasing the propagation angel.



(c) 90^0

Figure2 comparisions of the phase speed dispersion curves for unidirectional laminate by the LPM (the red line) with those solution obtained from RMM(the blue dotted)

4 CONCLUSIONS

Many different algorithms for solving guided waves propagation in composites layer have been investigated all these years, the article briefly analyzes them. Our work centre on the Legendre polynomial method (LPM) and the efforts for developing an algorithm of the advantages of robust. The Legendre method provide alternative to transfer matrix method. This paper discussed the mathematical formulation about Legendre polynomial method in detail. The cases of composite laminates have also been validated with the results obtained from TMM.

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