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Differential Quadrature Method Based Study of Vibrational Behaviour of Inclined Edge Cracked Beams

Shivani Srivastava and Raju Sethuraman

Department of Mechanical Engineering, IIT Madras, India

Abstract The study of vibration behaviour of cracked system is an important area of research. In the present work we present a mathematical model to study the effect of inclination, location and size of the crack on the vibrational behavior of beam with different boundary conditions. The model is based on the assumption that the equivalent flexible rigidity of the cracked beam can be written in terms of the flexible rigidity of the uncracked beam, based on the energy approach as proposed by earlier researchers. In the present work the Differential Quadrature Method (DQM) is used to solve equation of motion derived by using Euler's beam theory. The primary interest of the paper is to study the effect of inclined crack on natural frequency. We have also studied the beam vibration with and without vertical edge crack as a special case to validate the model. The DQM results for the natural frequencies of cracked beams agree well with other literature values and ANSYS solutions.

1 Introduction

The behavior of structures containing cracks is the interesting area of research in the light of potential developments in automatic monitoring of structure quality. The study of influence on eigen-frequencies and modes shapes of the structure due to crack is important in many aspects. A number of research has been reported their work in this area. A crack introduces a local flexibility in a system which is a function of crack depth. The dynamical behavior of the system and its stability characteristics changes due to the flexibility. Here, in this work, we have taken the beam structure, specifically the Bernoulli-Euler beam is of our interest with appropriate boundary conditions. In this paper, the assumption and formulation of the model has been discussed in Section 2. Section 3 deals with the methodology used to solve the governing equation. Case studies are reported in Section 4, and the concluding remarks are given in Section 5.

2 Model

The variation of the equivalent bending stiffness and depth (along the beam length) for a cracked beam are obtained using an energy-based model as proposed by Yang et.al. [1] to investigate the influence of cracks on structural dynamic characteristics during the vibration of a beam with open crack. transverse vibration are obtained for a rectangular beam containing cracks. Here, we have extended the model for inclined crack (Figure 1).

2.1 Assumptions for the model

- At the location of the crack the local stiffness got reduced due to crack.
- The change in strain energy due to crack under constant load assumption is computed using energy balance approach.
- The equivalent bending stiffness and equivalent depth of the beam is obtained by modeling strain energy variation along the beam length.
- Crack is always open during vibration.

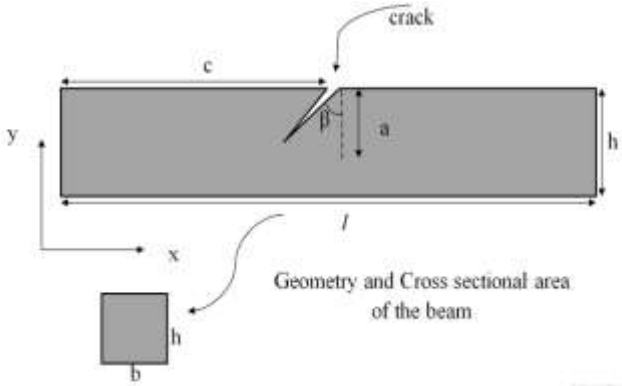

Figure 1. Geometry of the inclined edge cracked beam.

For an uncracked beam the strain energy is given by

$$U = \frac{1}{2}\int_0^l \frac{M^2}{EI}dx \qquad (2.1)$$

Energy needed for a crack growth of length *'a'* is given by

$$E_c = \int_0^a G dA = \int_0^a G b da \qquad (2.2)$$

where, G is the strain energy release rate.

2.1.1 Vertical edge crack

For vertical edge crack G is given as,

$$G = \frac{K_1^2}{E} \qquad (2.3)$$

where, K_I is the stress intensity factor of the first mode and is given by,

$$K_1 = K_{I(0)} = \frac{6M\sqrt{\pi a}}{bh^2} f(a) \qquad (2.4)$$

where, M is the bending moment of the beam, and

$$f(a) = 1.12 - 1.4\left(\frac{a}{h}\right) + 7.33\left(\frac{a}{h}\right)^2 - 13.8\left(\frac{a}{h}\right)^3 + 14\left(\frac{a}{h}\right)^4 \qquad (2.5)$$

where, $a/h < 0.6$

2.1.2 Inclined edge crack

Let the kink angle is defined by α and crack is assumed to be inclined on an angle β [2],

$$G = \frac{k_1^2(\alpha) + k_2^2(\alpha)}{E} \qquad (2.6)$$

where,

$$k_1(\alpha) = C_{11}K_I + C_{12}K_{II}$$
$$k_2(\alpha) = C_{21}K_I + C_{22}K_{II} \qquad (2.7)$$

where, k_1 and k_2 are the local stress intensity factor at the tip of the kink and K_I and K_{II} are the stress intensity factors for the main tilted crack given by equation (2.8-2.9), (figure(2))

$$K_I = K_{I(0)}\cos^2\beta$$
$$K_{II} = K_{I(0)}\cos\beta\sin\beta \qquad (2.8)$$

The coefficients C_{ij} are defined by,

$$C_{11} = \frac{3}{4}\cos\left(\frac{\alpha}{2}\right) + \frac{1}{4}\cos\left(\frac{3\alpha}{2}\right)$$
$$C_{12} = -\frac{3}{4}\left[\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{3\alpha}{2}\right)\right] \qquad (2.9)$$
$$C_{21} = \frac{1}{4}\left[\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{3\alpha}{2}\right)\right]$$
$$C_{22} = \frac{1}{4}\cos\left(\frac{\alpha}{2}\right) + \frac{3}{4}\cos\left(\frac{3\alpha}{2}\right)$$

Further, if EI_c is the bending stiffness of the cracked beam, strain energy in the cracked beam can be written as:

$$U_c = \frac{1}{2}\int_0^1 \frac{M^2}{EI_c} dx \qquad (2.10)$$

For the transverse vibration of the long beam, the crack is mainly subjected to the direct bending stresses and the shear stresses can be neglected; therefore, only the first mode crack exists. While the inclined crack includes both Mode I and Mode II (mixed type) crack. The model defined in section 2 can be defined for vertical edge crack by using the strain energy release rate as shown in equation (2.3). For inclined crack the equivalent stored energy G is the function of both K_I and K_{II} (equation(2.6)). Equation 2.16-2.19 gives the model of equivalent approach for vertical edge crack.

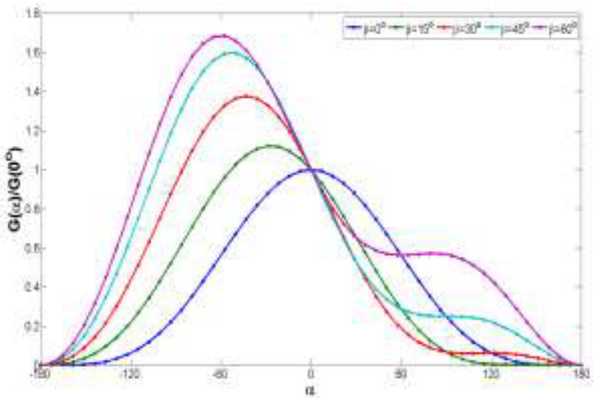

Figure 2. Effect of crack angle on strain energy [2].

A similar model has been used for the inclined edge crack with the modification in strain energy release rate as discussed above.

2.2 Energy expression

The stresses/strains are highly concentrated around the crack tip, and reach the nominal stress at a location far away from the crack. So it can be assumed that the increase of strain energy due to crack growth, under constant applied moment, is concentrated mainly around the crack region. The energy consumed for crack growth along the beam defined by [3-4], equation1, (figure (3)).

$$E_c = A(a,c)\exp\{-|B(x)/C(a)|\} \qquad (2.11)$$

for the continuity of the function let, $|B(x)/C(a)| = y^2$ where, $y^2 = ((x-c)/k(a))^2$, which gives,

$$E_c = A(a,c)\exp\{-((x-c)/k(a))^2\} \quad (2.12)$$

on expanding the exponential series, and neglecting the $O(h^2)$ terms, we get the final energy function form as,

$$E_c = A(a,c)/\{1+((x-c)/k(a))^2\} \quad (2.13)$$

where, the terms *A(a,c)* and *k(a)* are determined over the beam length for which

$$E_c = \int_0^l A(a,c)/\{1+((x-c)/k(a))^2\}\,dx \quad (2.14)$$

where, *c* is location of the crack from one end of the beam. Now with the help of equations 2.2-2.5 and 2.14, we can obtain,

$$A(a,c) = \frac{D(a)M^2}{k(a)\left[\tan^{-1}\left(\frac{l-c}{\{k(a)\}}\right) + \tan^{-1}\left(\frac{c}{\{k(a)\}}\right)\right]} \quad (2.15)$$

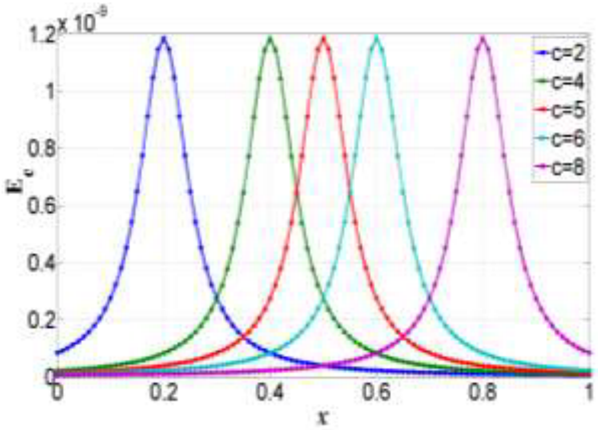

Figure 3. Variation of energy function with crack location for *a=0.8*.

and by using 2.1, 2.10 and 2.14, together with the assumption that the final strain energy in the cracked beam is $U_c = U + E_c$, we get the following relations:

Modified bending stiffness and height (figures 4and 5) of the beam is given by:

$$EI_c = \frac{EI}{1 + \frac{EIR(a,c)}{\left[1+\left(\frac{x-c}{\{k(a)\}}\right)^2\right]}} \quad (2.16)$$

$$R(a,c) = \frac{2D(a)}{k(a)\left[\tan^{-1}\left(\frac{1-c}{\{k(a)\}}\right) + \tan^{-1}\left(\frac{c}{\{k(a)\}}\right)\right]} \quad (2.17)$$

$$D(a) = \frac{18\pi[f(a)^2]a^2}{Ebh^4} \quad (2.18)$$

$$k(a) = \frac{3\pi[f(a)^2](h-a)^3 a^2}{[h^3-(h-a)^3]h} \quad (2.19)$$

Also, at the position when *x=c,* we have the following relation:

$$\frac{EI}{EI_c} = \frac{h^3}{(h-a)^3} \quad (2.20)$$

2.3 Equation of motion

For an Euler beam the transverse vibration equation of cracked beam is given by,

$$\frac{\partial^2}{\partial x^2}\left[EI_c\frac{\partial^2 w}{\partial x^2}\right] + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \quad (2.21)$$

where, ρ is density and *A* is the cross-sectional area of the beam.

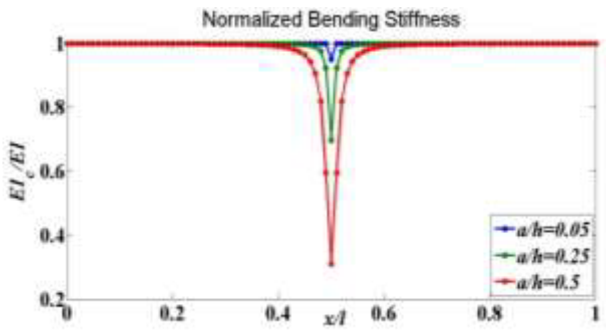

Figure 4. Equivalent bending stiffness *for c/l=0.5, l=10, b=1,h=1.*

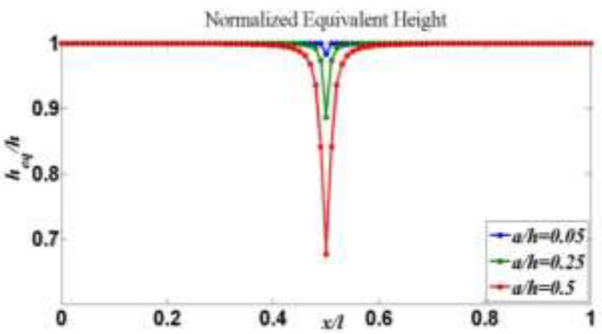

Figure 5. Equivalent height variation for *c/l=0.5 l=10, b=1,h=1.*

Let, *w(x,t)=W(x)exp(iωt)* , equation (2.21) becomes

$$\frac{d^2}{dx^2}\left[EI_c\frac{d^2W}{dx^2}\right]-\rho A\omega^2W=0 \quad (2.22)$$

where, EI_c is given by (2.16). Equation (2.22) can be rewritten as,

$$EI_c\frac{d^4W}{dx^4}+2\left(\frac{dEI_c}{dx}\right)\frac{d^3W}{dx^3}+\left(\frac{d^2EI_c}{dx^2}\right)\frac{d^2W}{dx^2}-\rho A\omega^2W=0 \quad (2.23)$$

3 Methodology (DQM)

In this section a brief overview of DQM has been discussed [5]. It is assumed that a function *W(x)* is sufficiently smooth over the whole domain. The n^{th} order derivative of the function *W(x)* with respect to *x* at *m* number of grid points x_i, is approximated by a linear sum of all the functional values in the whole domain, that is,

$$W_x^{(n)}(x_i)=\sum_{j=1}^{m}c_{ij}^{(n)}W(x_j),\ i=1,23,...,m; \quad (3.1)$$

where, c_{ij} represent the weighting coefficients, and *n* is the number of grid points in the whole domain. Equation 1 is called differential quadrature (DQ). It should be noted that the weighting coefficients c_{ij} are different at different locations of x_i. The weighting coefficients required in DQ method as shown in equation(1) are defined recursively by equations(3.2-3.5).

$$c_{ij}^{(n)}=n\left(c_{ii}^{(n-1)}c_{ij}^{(n-1)}-\frac{c_{ij}^{n-1}}{x_i-x_j}\right) \text{ for } i, j=1, 2, 3, ..., m;$$
$$j\neq i;\ n=2,3,4. \quad (3.2)$$

$$c_{ii}^{(n)}=-\sum_{\substack{j=1\\ j\neq i}}^{m}c_{ij}^{(n)},\ i=1, 2, 3, ..., m; \quad (3.3)$$

$$c_{ij}^{(1)}=\frac{M^{(1)}(x_i)}{(x_i-x_j)M^{(1)}(x_j)},\quad i, j=1, 2,..., m;\ i\neq j \quad (3.4)$$

$$M^{(1)}(x_i)=\prod_{\substack{j=1\\ j\neq i}}^{m}(x_i-x_j) \quad (3.5)$$

3.1 Grid point Distribution

The selection of number and type of grid points has a significant effect on the accuracy of the DQM results. It is found that the optimal selection of the sampling points in the vibration problems is the normalized Chebyshev-Gauss-Lobatto points [5].

4 Case Studies

The Model discussed in this paper has been used for different cracks and boundary conditions. The governing equation of motion (equation(2.23)) has been transformed into a discrete eigen value problem with the help of DQM as given in equations 3.1-3.5. Also the ANSYS models has been used to compare the solution. Figure 6 shows an ansys model of actual vertical edge cracked beam, while figure 7 depicts the equivalent cracked model of the beam. The inclined edge cracked model of the beam is shown in Figure 8. Table 1-2 shows the numerical results obtained for uncracked beams by using our model as a special case. While in table 3 the effect of crack location on frequency has been shown. Tables 4 and 5 compares the effect of crack angle on frequency parameters for clamped and fixed boundary conditions. In table 6 the effect of orientation of crack has been shown for clamped beam.

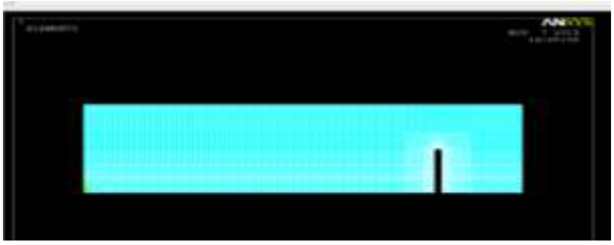

Figure 6. Vertical U-notch crack in beam ANSYS.

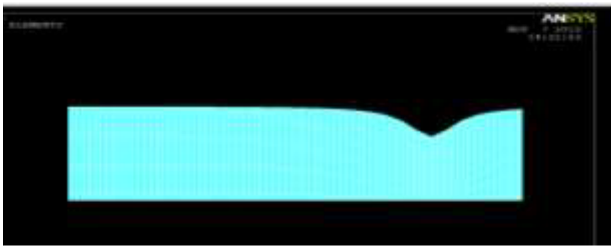

Figure 7. Equivalent vertical edge cracked in beam ANSYS.

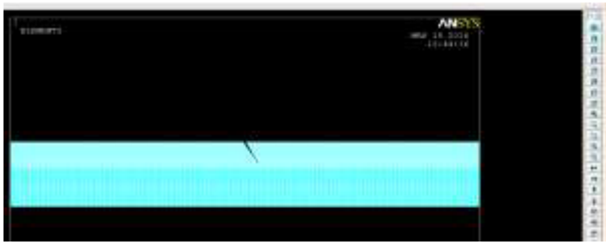

Figure 8. V-notch Inclined edge cracked in beam (45O) ANSYS.

Table 1. Comparison of frequency parameter for uncracked simply-supported beam for *E=2.16GPa, ρ=7650kg/m³, l=0.4m, b=h=0.01m.*

Frequency (Hz)	Experimental	ANSYS				DQM
	Ref.[6]	20 Node Brick Element	Beam Element 2D	Plane Stress with Thickness	Plane stress with υ=0.3	15 Grid Point
I	151.5	150.44	150.55	150.46	157.69	150.593
II	602.5	599.92	601.75	600.26	628.69	602.373

Table 2. Comparison of frequency parameter for uncracked cantilever beam for *E=70GPa, ρ=2710kg/m*^{3}*, l=800mm, b=60mm, h=6mm*

Frequency (Hz)	Experimental	ANSYS				DQM
	Ref.[7]	20 Node Brick Element	Beam Element 2D	Plane Stress with Thickness	Plane stress with υ=0.0	10 Grid Point
I	8.0217	7.7486	7.6969	7.7003	7.6966	7.6969
II	50.256	48.539	48.232	48.245	48.223	48.3753
III	141.13	135.92	135.04	135.03	134.98	139.4364

Table 3. Effect of crack location on vertically edge cracked simply-supported beam for *E=2.16GPa, ρ=7650kg/m*^{3}*, l=0.4m, b=h=0.01m.*

Frequency (Hz)	*a/h*=0.1	Experimental Ref.[6]	DQM 26 pts
I	*c/l*=0.2	151.2	150.49
II		599.8	597.42
I	*c/l*=0.3	150.5	150.59
II		604.2	601.03
I	*c/l*=0.5	149.2	147.17
II		599.8	599.21

Table 4. Comparison of frequency parameter for inclined (30^O) edge cracked cantilever beam for *E=70GPa, ρ=2710kg/m*^{3}*, l=800mm, b=60mm h=6mm, c/l=0.40, a/h=0.20.*

Mode	ANSYS		Experimental [7]	FEA	DQM
	υ=0.0	υ=0.346			
I	7.6799	7.6795	7.8200	7.6925	7.5137
II	47.282	47.061	49.5900	48.542	51.5731
III	128.25	126.87	138.41	135.68	136.371

Table 5. Effect of crack inclination on frequency parameter for fixed beam for *E=2GPa, ρ=7850kg/m*^{3}*, l=10m, b=h=1m, c=5.0, a=0.4 υ=0.3.*

Angle	Actual Crack Model ANSYS			Equivalent Crack Model ANSYS			DQM Model		
	I	II	III	I	II	III	I	II	III
30°	7.8140	42.168	129.60	8.0250	47.193	129.89	7.9591	43.4381	122.493
45°	7.7651	42.004	120.63	7.8393	41.862	119.66	6.7952	41.9609	118.747
60°	7.7110	41.526	129.28	8.1002	48.903	129.91	7.3828	45.8426	129.004

Table 6. Effect of crack orientation about the crack tip on frequency in fixed beam for *E=2GPa, ρ=2700kg/m*^{3}*, l=10m, b=h=1m, c/l=0.5, a/h=0.4 υ=0.3.*

Angle	Left side inclination			Right side inclination		
	I	II	III	I	II	III
0°	4.1918	22.803	69.921	4.1918	22.803	69.921
15°	4.1895	22.761	69.909	4.1889	22.758	69.915
30°	4.1892	22.737	69.882	4.1873	22.736	69.898
45°	4.1870	22.649	69.822	4.1806	22.646	69.850
60°	4.1297	20.664	67.677	4.1578	22.391	69.710
75°	4.0963	22.525	67.156	4.0518	21.359	68.808

5 Conclusion

The model discussed in this paper has been used to find the effect of inclined crack on beam vibrations. The governing equation of motion is solved by using Differential Quadrature Method using Chebyshev's collocation points. It has been observed that the model gives optimum results for different types of boundary conditions. The applicability of the model has been obtained by the comparing the results with ANSYS actual crack model and ANSYS equivalent crack model for two types of crack (i) inclined crack and (ii) vertical crack. The results are also compared with the uncracked beam. The comparative study justify and validate the model.

References

1. X. F. Yang, A. S. J. Swamidas, R. Seshadri, Crack Identification in Vibrating Beams using The Energy Method. *J Sound Vib*., **244**, 339-357 (2001).
2. T.L. Anderson, Fracture Mechanics Fundamental and Application, Third edition, CRC Press, Taylor and Francis Group, (2013).
3. S. Christides, A.D.S. Barr, One-Dimensional Theory of Cracked Bernoulli-Euler Beams. Int. J. Mech. Sci., **26(11/12)**, 639-648 (1984).
4. M.I. Friswell, J.E.T. Penny, Crack Modeling for Structural Health Monitoring. Struct Heal Monit., **1(2)**, 139-148 (2002).
5. C. Shu, Differential Quadrature and Its Applicationin Engineering, Springer, London, (2000).
6. H. Yoon, I. N. Son, S. J. Ahn, Free Vibration Analysis of Euler-Bernoulli Beam with Double Cracks, Journal of mechanical science and technology, **21**, 476-485 (2007).
7. R.K. Behera, A. Pandey, D.R. Parhi, Numerical and Experimental Verification of a Method for Prognosis of Inclined Edge Crack in Cantilever Beam Based on Synthesis of Mode Shapes. *Procedia Technol*, **14,** 67-74 (2014).