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An analysis of gearing

Tadeusz Nieszporek[1], *Piotr* Boral[1], and *Rafał* Gołębski[1,*]

[1]Czestochowa University of Technology, Institute for Mechanical Technologies, Al. Al. Armii Krajowej 21, 42-201 Czestochowa, Poland

Abstract. The article has presented a numerical method for determining the gear ratio function and the gearing line of the toothed gear. The described algorithm, explained on the example of planar gearing, can be used also for the analysis of spatial gearings. Assembly errors have been allowed for in the gear model.

1 Introduction

The quality of a gear is determined by the analysis of the gearing [1-3], the key element of which is the determination of the gear ratio function [4]. The gearing analysis is used chiefly for determining the kinematic errors that result from toothing cutting errors and gear assembly errors, as well as for optimizing the gear synthesis (especially for the analysis of spatial gearing) [5, 6]. Generally, this is a process involving numerical computation by successive approximation methods.

The method of planar gearing analysis was first formulated by Litvin (1968) [4]. This method can be used not only for toothed wheels and gears, but also for the majority of simple mechanisms (such as cam mechanisms). A similar discussion as for the planar gearing can be conducted for the spatial gearing. The gearing analysis, as supplemented by the analysis of the gear contact trace, is called Tooth Contact Analysis (TCA) [4]. Programs for spatial TCA were first developed by Gleason Works (1960) and by Litvin and Gutman (1981).

It is assumed that the tooth profiles are given, and the distance between the gear wheel axes is also given. The gear ratio function (relationship between gear wheel rotations) and the gearing lines need to be determined.

2 Conditions for tooth profile tangency

Coordinate systems S_1, S_2 and S_f are introduced, which are rigidly tied, respectively, with gear wheels *1* and *2* and with the gear casing. In the case of a planar gearing, the tooth profiles Σ_1 and Σ_2 (of gear wheels *1* and *2*) and the unit vectors of the normals to the profiles are equal [4] - Figure 1.

[*] Corresponding author: rafal@itm.pcz.pl

$$\left\{\theta_1 \in \langle a_1, \quad b_1 \rangle \,:\, \mathbf{r}_1^{(1)} = \mathbf{r}_1^{(1)}(\theta_1) \in C^2 \right\} \quad \frac{\partial\, \mathbf{r}_1^{(1)}}{\partial \theta_1} \neq 0 \tag{1}$$

$$\mathbf{n}_1^{(1)} = \mathbf{n}_1^{(1)}(\theta_1) \tag{2}$$

$$\left\{\theta_2 \in \langle a_2, \quad b_2 \rangle \,:\, \mathbf{r}_2^{(2)} = \mathbf{r}_2^{(2)}(\theta_2) \in C^2 \right\} \quad \frac{\partial\, \mathbf{r}_2^{(2)}}{\partial \theta_2} \neq 0 \tag{3}$$

$$\mathbf{n}_2^{(2)} = \mathbf{n}_2^{(2)}(\theta_2) \tag{4}$$

where: θ_1, θ_2 - parameters of the tooth profiles Σ_1 and Σ_2 (of the gear wheels *1* and *2*).

The superscripts (1) and (2) at the vectors refer to the tooth profiles Σ_1 and Σ_2, while the subscripts indicate the coordinate system, in which these quantities are written.

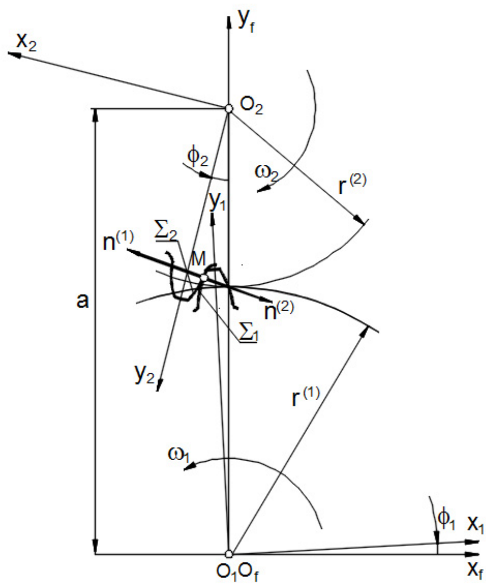

Fig. 1. The coordinate systems and gear wheel tooth profiles.

The transformation of the vector coordinates and the normal unit vector components from the coordinate systems S_1 and S_2 to the stationary reference system S_f can be written as follows [7, 8]

$$\mathbf{r}_f^{(i)}(\theta_i, \quad \phi_i) = \left[M_{fi}(\phi_i)\right]\mathbf{r}_i^{(i)}(\theta_i) \qquad i = 1,2 \tag{5ab}$$

$$\mathbf{n}_f^{(i)}(\theta_i, \quad \phi_i) = \left[L_{fi}(\phi_i)\right] \, \mathbf{n}_i^{(i)}(\theta_i) \tag{6ab}$$

where: ϕ_1, ϕ_2 - rotation angles of gear wheels *1* and *2*.

The matrix $[M]$ (with a dimension of 4 x 4) considers the mutual displacement and rotation of the coordinate systems, while the matrix $[L]$ (with a dimension of 3 x 3) considers only the rotation.

The tangency of the profiles occurs only when the profiles Σ_1 and Σ_2 contact at the same point and their normal unit vectors are collinear (parallel) at that point. Hence

$$\mathbf{r}_f^{(1)}(\theta_1, \quad \phi_1) = \mathbf{r}_f^{(2)}(\theta_2, \quad \phi_2) \qquad \mathbf{n}_f^{(1)}(\theta_1, \quad \phi_1) = \mathbf{n}_f^{(2)}(\theta_2, \quad \phi_2) \tag{7ab}$$

3 Gearing analysis

The components of the unit vectors $\mathbf{n}_f^{(1)}$ and $\mathbf{n}_f^{(2)}$ for the planar gearing (with the components $n_{fz}^{(1)} = n_{fz}^{(2)} = 0$) meet the condition

$$n_{fx}^{(1)} = n_{fx}^{(2)} \qquad \Rightarrow \qquad n_{fy}^{(1)} = n_{fy}^{(2)} \tag{8ab}$$

where: $n_{f*}^{(i)}$ - component $*$ of the unit vector i ($i = 1{,}2$) in the system S_f.

Therefore, the projections of the position vectors (7a) and the normal unit vectors (7b) onto the axes of the coordinate system S_f yield four scalar equations, of which only three are independent. Equations (7) i (8) give a system of three equations with four unknowns [4]

$$f_j(\theta_1, \quad \phi_1, \quad \theta_2, \quad \phi_2) = 0 \qquad j = 1, ..., 3 \tag{9a-c}$$

where

$$\{f_1, \quad f_2, \quad f_3\} \in C^1 \tag{10}$$

The analysis of the planar gearing will be defined, if equations (10) can be transformed into three equations as a function of one variable

$$\{\, \theta_1(\phi_1), \quad \theta_2(\phi_1), \quad \phi_2(\phi_1) \,\} \in C^1 \tag{11}$$

The function $\phi_2(\phi_1)$ represents the relationship between gear wheel rotation angles, or the gear ratio function. The functions $\theta_1(\phi_1)$ and $\theta_2(\phi_1)$ define the points on the profiles Σ_1 and Σ_2, at which the profiles are tangent for a given rotation angle value of gear wheel *1*. The gearing line of the profiles Σ_1 and Σ_2 is described by the function

$$\mathbf{r}_f^{(1)} = \mathbf{r}_f^{(1)}(\theta_1(\phi_1), \quad \phi_1) \quad \vee \qquad \mathbf{r}_f^{(2)} = \mathbf{r}_f^{(2)}(\theta_2(\phi_1), \quad \phi_2(\phi_1)) \tag{12ab}$$

4 The analytical algorithm

Litvin assumes [4] that a starting point of the iteration $P^{(0)} = \left(\theta_1^{(0)}, \quad \phi_1^{(0)}, \quad \theta_2^{(0)}, \quad \phi_2^{(0)}\right)$ is given, which satisfies equations (9). In the next step, the successive point is determined, which satisfies equations (9)

$$P^{(1)} = \left(\theta_1^{(1)}, \quad \phi_1^{(1)}, \quad \theta_2^{(1)}, \quad \phi_2^{(1)}\right) \tag{13}$$

where the superscript denotes the successive approximation.

A system of two equations is chosen from the system of equations (9). It is assumed that the following equations are chosen:

$$f_2\left(\theta_1,\quad \phi_1,\quad \theta_2,\quad \phi_2\right)=0 \qquad\qquad f_3\left(\theta_1,\quad \phi_1,\quad \theta_2,\quad \phi_2\right)=0 \tag{14ab}$$

Taking $\phi_1=\phi_1^{(1)}$ as given, it should be assumed that $\theta_1=\theta_1^{(1)}$. In view of the above, the system of equations (14) can be solved with respect to θ_2 and ϕ_2. Let the solution be $\theta_2^{(1)}$ and $\phi_2^{(1)}$. The new set of parameters (13) is defined, if it satisfies the remaining equation

$$f_1\left(\theta_1,\quad \phi_1,\quad \theta_2,\quad \phi_2\right)=0 \tag{15}$$

by the set of parameters $\left(\theta_1^{(1)},\quad \phi_1^{(1)},\quad \theta_2^{(1)},\quad \phi_2^{(1)}\right)$. If equation (15) is not satisfied, then a new value should be taken for $\theta_1^{(1)}$ and the system of equations (14) should be solved again, and then the condition (15) needs to be rechecked.

The above algorithm involves splitting the system (9) of three equations into two subsystems, one of them containing two equations and the other one, and solving them independently.

5 The numerical algorithm

The parameters ϕ_1 and ϕ_2 are the angles of rotations of gear wheels *1* and *2*, while the parameters θ_1 and θ_2 are the parameters of the gear wheel profiles Σ_1 and Σ_2.

In the reference system Σ_f, in the system of coordinates $\left(x_f \quad y_f\right)$, a grid of ordinate lines with fixed values of the abscissae x_{fl} $\left(l=1,..,m\right)$, uniformly arranged (with a fixed step), is introduced - Figure 2a (with unrealistic values of the ordinates $y_{f\,\max}$). For the fixed value of parameter $\phi_1=\phi_{1r}$ $\left(r=1,..,t\right)$, from the profile equation Σ_1

$$\mathbf{r}_f^{(1)}=\mathbf{r}_f^{(1)}\left(\theta_1,\quad \phi_1\right) \tag{16}$$

for successive values of the coordinates of parameter $\theta_1=\theta_{1j}$ $\left(j=1..n\right)$, the points of the profile Σ_1 are determined in the system of coordinates $\left(x_f \quad y_f\right)$. In doing so, values of parameter θ_1 can be taken uniformly with a fixed step within the assumed interval. The profile points, in a general case, do not lie on the ordinate lines with the abscissae x_{fl} of the system $\left(x_f \quad y_f\right)$ – Figure 2b. As the grid lines should intersect with the profile only once, the grid is taken in the computation program either on the axis of abscissae – Figure 2 or on the axis of ordinates, respectively.

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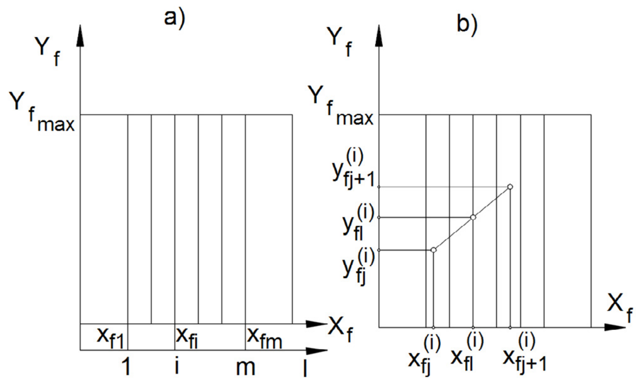

Fig. 2. The grid of abscissae: a) and the gear wheel tooth profile, b) in the system $(x_f \quad y_f)$.

The ordinates of the line segment points lying on the ordinate lines are determined from the following relationship – Figure 2b

$$y^{(i)}_{fl} = y^{(i)}_{fj} + \frac{y^{(i)}_{fj+1} - y^{(i)}_{fj}}{x^{(i)}_{fj+1} - x^{(i)}_{fj}} \left(x^{(i)}_{fl} - x^{(i)}_{fj} \right) \qquad i = 1,2 \tag{17}$$

Substituting for x_f the values of the abscissae of the grid ordinate lines lying within a given segment (or intersecting the segment under consideration), we will obtain the points of the curve on those ordinate lines (the points of grid ordinate line intersection with the analysed segment).

By joining successive profile points with line segments and determining the points of intersection of those segments with the ordinate lines with the abscissae x_{fl} in the system $(x_f \quad y_f)$, the profile Σ_1 is substituted with a set of ordered (normalized) points.

The profile Σ_2 should be dealt with in a similar manner. For the fixed value of parameter $\phi_2 = \phi_{2s}$ $(s = 1,..,v)$, from the profile equation Σ_2

$$\mathbf{r}^{(2)}_f = \mathbf{r}^{(2)}_f \left(\theta_2, \quad \phi_2 \right) \tag{18}$$

for successive values of the coordinates of parameter $\theta_2 = \theta_{2k}$ $(k = 1..p)$, the points of the profile Σ_2 are determined in the system of coordinates $(x_f \quad y_f)$.

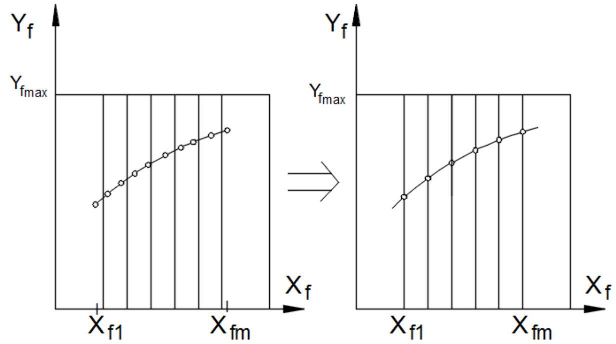

Fig. 3. The arrangement of profile points.

By joining successive profile points with line segments and determining the points of intersection of those segments with the ordinate lines with the abscissae x_{fl} in the system $\left(x_f \quad y_f\right)$, the profile Σ_2 is substituted with a set of ordered points – Figure 3.

By comparing the ordinates of the points of the profiles Σ_1 and Σ_2 for the same abscissae x_{fi}, the distances of those points are determined – Figure 4

$$\Delta_{fl} = y_{fl}^{(2)} - y_{fl}^{(1)} \tag{19}$$

where the subscript f identifies the system of coordinates, while the subscript l, the ordinate (with the abscissa x_{fi}). At the point of contact (on condition that the profiles do not interpenetrate at other points) the distance is $\Delta_{fl} = 0.0$ (with the assumed computation accuracy). If this condition is not met, the value of angle $\phi_2 = \phi_{2s}$ should be changed, each time (for successive values of this parameter) repeating the computation process to determine the profile contact point. With a fixed value of ϕ_1, the value of parameter ϕ_2 is determined by the step halving method [8].

The entire computation process should be repeated for successive values of angle $\phi_1 = \phi_{1r}$ to ultimately determine the gear ratio function $\phi_2\left(\phi_1\right)$.

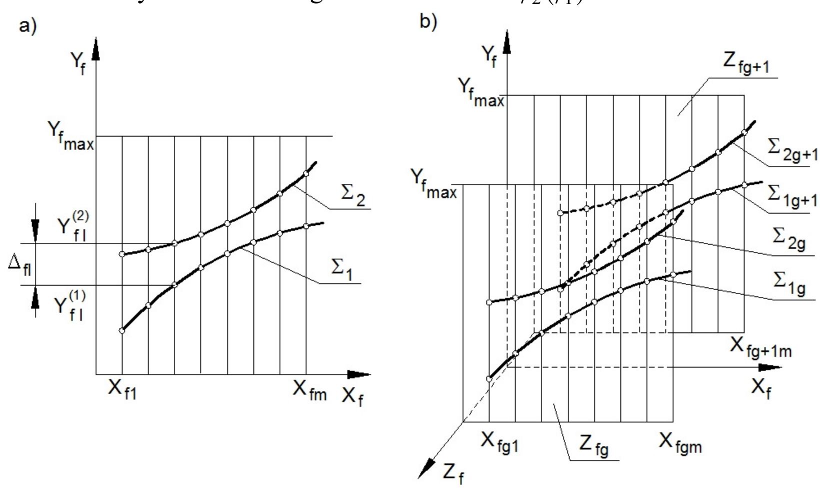

Fig. 4. Determination of the gear wheel tooth contact point.

The algorithm described above is simple, which constitutes its great asset. It can also be used for the analysis of spatial gearing.

In the case of spatial gearing analysis, that is the analysis of toothing surface mating, the surfaces should be determined as sets of profiles in parallel cutting planes – Figure 4b. The above-described algorithm should be repeated for successive cutting planes z_{fg} $\left(g = 1,..w\right)$. The surface equations can be written as

$$\mathbf{r}_i^{(i)} = \mathbf{r}_i^{(i)}\left(\theta_i, \quad z_{ig}\right) \quad i = 1,2 \tag{20}$$

where z_{ig} defines the position of the cutting plane, while being also a surface parameter.

The surface equations in the reference system Σ_f are as follows

$$\mathbf{r}_f^{(1)} = \mathbf{r}_f^{(i)}\left(\theta_i, \quad \phi_i, \quad z_{ig}\right) \;\; i = 1{,}2 \tag{21}$$

By comparing the ordinates of surface points for the same abscissae x_{fgi} (in a given cutting plane), the distances of those points are determined

$$\Delta_{fgl} = y_{fgl}^{(2)} - y_{fgl}^{(1)} \tag{22}$$

where the subscript g identifies the cutting plane, while the subscript l, the ordinate (with the abscissa x_{fgi}) in that plane.

As a result, the surfaces will be determined in the form of sets of normalized points lying on ordinate lines in cutting planes, which enables the examination of their mutual position and determination of the gear ratio function. Whereas, the surfaces may only contact linearly or be tangent in one point only. Therefore, for a fixed value of angle ϕ_{1r}, a value of angle ϕ_{2s} should be determined (by the successive approximation method) such, that in individual cutting planes there will be $\Delta_{fgl} > 0.0$, and at least in one plane and at one point, there will be $\Delta_{fgl} = 0.0$.

6 Example

A spur gear with an involute tooth profile is taken – Figure 5.

The following is assumed: module, $m = 5\,[mm]$; the number of the teeth of gear wheel *1*, $z_1 = 30$; the number of the teeth of gear wheel *2*, $z_2 = 20$; profile angle, $\alpha_0 = 20$ [deg]; axis distance error, $e = -1, 0, 1\,[mm]$; profile modification depth, $t_m = 2\,[mm]$; profile modification length, $g_m = 1\,[mm]$.

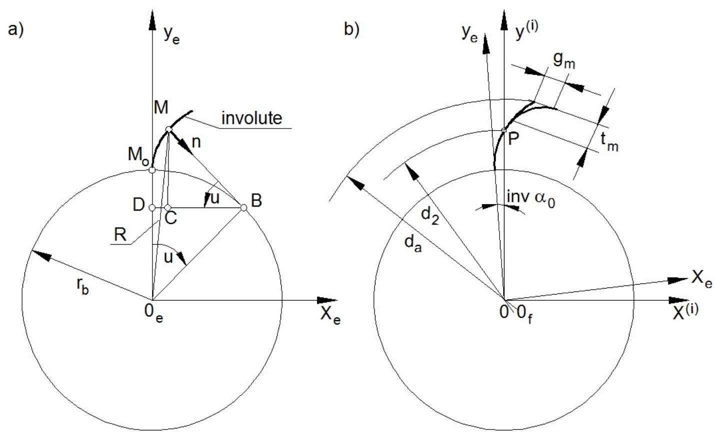

Fig. 5. The profile: a) involute, b) of the gear wheel.

Figure 5a shows that the profile of the involute and the unit vector normal to the profile can be written as

$$\mathbf{r}_e = r_b\left[su - ucu, \quad cu + u\,su\right] \tag{23}$$

$$\mathbf{n}_e = \left[cu, \quad -su\right] \tag{24}$$

where the following designations are taken

$$r_b = \frac{z\,m}{2} c\alpha_0 \qquad c* = \cos * \qquad s* = \sin* \tag{25a-c}$$

As follows from Figure 5b, the gear wheel tooth profile and the unit vector of the normal to the profile can be written as

$$\mathbf{r}_i^{(i)} = r_{bi}\left[s(u - inv\alpha_0) - u\, c(u - inv\alpha_0), \quad c(u - inv\alpha_0) + u\, s(u - inv\alpha_0) \right] \qquad i = 1,2 \tag{26}$$

$$\mathbf{n}_i^{(i)} = \left[c(u - inv\,\alpha_0), \quad -s(u - inv\,\alpha_0) \right] \tag{27}$$

where

$$inv\alpha_0 = tg\alpha_0 - \alpha_0 \tag{28}$$

The matrices of transition from the gear wheel coordinate systems to the stationary coordinate system linked with the gear casing are as follows – Figure 1

$$\left[M_{f1}\right] = \begin{bmatrix} c\phi_1 & -s\phi_1 & 0 & 0 \\ s\phi_1 & c\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[L_{f1}\right] = \begin{bmatrix} c\phi_1 & -s\phi_1 & 0 \\ s\phi_1 & c\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29ab}$$

$$\left[M_{f2}\right] = \begin{bmatrix} -c\phi_2 & -s\phi_2 & 0 & 0 \\ s\phi_2 & -c\phi_2 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left[L_{f2}\right] = \begin{bmatrix} -c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & -c\phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30ab}$$

where the distance of the axis is equal to

$$a = (z_1 + z_2)\, m / 2.0 + e \tag{31}$$

Based on relationships (5) and (6), we can write:

$$\mathbf{r}_f^{(1)} = r_{b1}\left[s\beta - u\, c\beta, \quad c\beta + u\, s\beta \right] \qquad \mathbf{n}_f^{(1)} = \left[c\beta, \quad -s\beta \right] \tag{32ab}$$

$$\mathbf{r}_f^{(2)} = \left[r_{b2}(-s\gamma + u\, c\gamma), \quad r_{b2}(-c\gamma - u\, s\gamma) + a \right] \qquad \mathbf{n}_f^{(2)} = \left[-c\gamma, \quad s\gamma \right] \tag{33ab}$$

where the following designation is taken

$$\beta = u - inv\alpha_0 - \phi_1 \qquad \gamma = u - inv\alpha_0 + \phi_2 \tag{34ab}$$

where: r_{b1}, r_{b2} - base circle radii.

Therefore, the equations to be used for further computation are as follows:
1 - analytical method

$$f_2 = r_{b1}(c\beta + u\, s\beta) - r_{b2}(-c\gamma - u\, s\gamma) - a = 0 \tag{35}$$

$$f_3 = c\beta + c\gamma = 0 \tag{36}$$

$$f_1 = r_{b1}\left(s\beta - u\,c\beta\right) - r_{b2}\left(-s\gamma + u\,c\gamma\right) = 0 \tag{37}$$

2 - numerical method

$$\mathbf{r}_f^{(1)} = r_{b1}\left[s\beta - u\,c\beta, \quad c\beta + u\,s\beta\right] \tag{38}$$

$$\mathbf{r}_f^{(2)} = \left[r_{b2}\left(-s\gamma + u\,c\gamma\right), \quad r_{b2}\left(-c\gamma - u\,s\gamma\right) + a\right] \tag{39}$$

The gear ratio function is a linear function, which means that the gear ratio is constant. The error of the gear wheel axis distance has caused a parallel shift in the gear ratio function – Figure 6a. On the other hand, the modification of the profile by either rounding off or truncating the tooth tip – Figure 7, affects the gear ratio function – Figure 6b. The result is a gear ratio error and uneven gear operation.

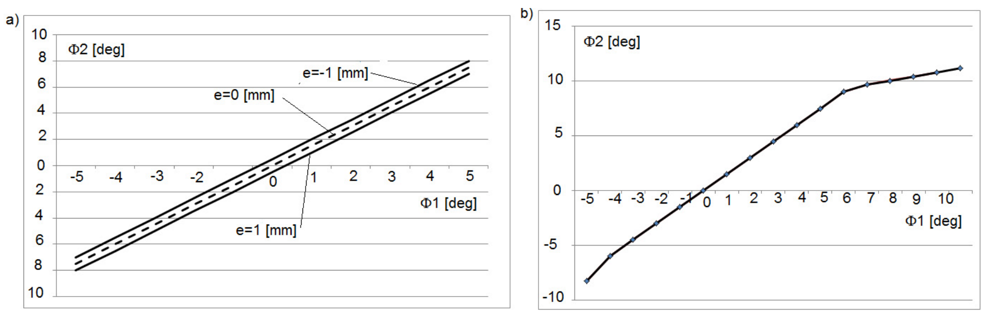

Fig. 6. The gear ratio function: a) with no gear wheel tooth profile modification – with allowance being made for the axis distance error; b) with tooth profile top modification.

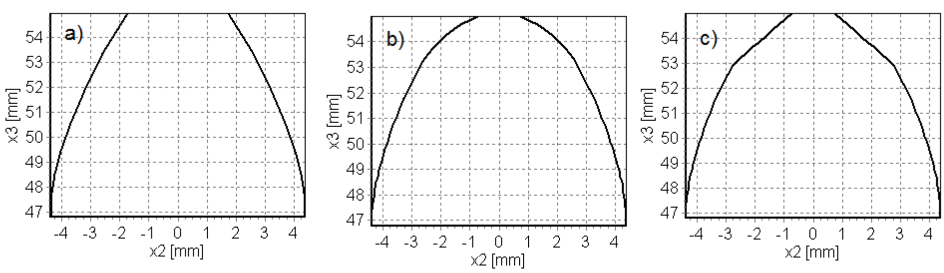

Fig. 7. The tooth profile of gear wheel 1 - a) without modification, b) with modification along the circular arc, c) with modification along the straight line.

7 Conclusions

The presented algorithm does not require the solving of systems of equations, in contrast to the analytical method using the Newtonian method for solving the system of equations. There is no need for allowing for the condition of normal unit vector collinearity, either.

The program consists of a module, in which the user enters profiles to be examined, and a universal module, in which the profile contact points are determined.

The effectiveness of the Newtonian method depends largely on the accuracy of the first approximations of roots, while in the proposed method, root variability ranges can be taken with a greater approximation.

The developed algorithm and computation program can also be used for the analysis of spatial gearing.

The computation accuracy depends on the number of iteration steps, the number of profile points and the number of grid lines, and can be easily changed.

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