Method of using of the Box-Cox transformation at the application of the xbar and s chart

Nicolae Eftimie^{1,*}

¹Transilvania University of Brasov, Department of Manufacturing Engineering, Mihai Viteazu No.5, Brasov, Romania

Abstract. The application of the most statistical process control techniques is based on the assumption that the distribution of the measurements is normal. However, there are many situations in practice when the process data distribution is not normal. In certain cases, the Box-Cox transformation can be used for converting the process data distribution into a normal distribution. Considering these aspects, the paper presents a method of application for the xbar and s chart that can be used in the case when the measurements distribution is not normal. The proposed method consists of the following stages: the testing of normality of the process data, the application of the Box-Cox transformation and the testing of normality of the transformed data. In the case when the distribution of the transformed data is normal, they are used at the application of the xbar and s control chart.

1 Introduction

The most of control charts that are used in practice are based on the assumption that the measurements distribution is normal. A tool often used in practice for testing this assumption is the Anderson-Darling normality test. A description of the procedure of application for this test is given in [1].

However in practice there are situations when the measurements are not normally distributed. One approach that can be applied in these cases is the transformation of the process data in order to obtain a set of normally distributed values [2].

Box and Cox [3] developed a method that can be used for converting the process data distribution into a normal distribution. According to this method, each measurement x_i will be transformed by means of the following system of equations:

$$y_i = \begin{cases} x_i^\lambda & \lambda \neq 0 \\ \ln(x_i) & \lambda = 0 \end{cases} \quad (1)$$

The λ parameter is determined as follows:

- The standardized values of the transformed data are calculated using the system of equations (2):

* Corresponding author : eftimien@unitbv.ro

$$z_i = \begin{cases} \frac{x_i^\lambda - 1}{\lambda \cdot G^{\lambda-1}} & \lambda \neq 0 \\ G \cdot \ln(x_i) & \lambda = 0 \end{cases} \quad (2)$$

where G represents the geometric mean of the process data.

- It is calculated the standard deviation for the distribution of the z_i variable, by means of the equation (3):

$$\hat{\sigma}_z = \frac{\bar{s}_z}{c_4} \quad (3)$$

where the c_4 coefficient is computed using the relation (4) [2]:

$$c_4 = \sqrt{\frac{2}{n-1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \quad (4)$$

where n is the size of the sample.

- It is determined the value of λ as being that for which the standard deviation of the z_i variable distribution is minimal.

2 Problem formulation

In the case when the measurements distribution is not normal, the procedure of application for the $\bar{x}$ and s chart becomes difficult.

The objective of the paper is to develop a method of application for this type of control chart in the cases when the distribution of the process data can be converted into a normal distribution by means of the Box-Cox transformation.

The method consists of the following stages:

- The testing of normality of the process data. If the measurements distribution is normal, they can be used at the control chart application, otherwise the data must be transformed.
- The application of the Box-Cox transformation.
- The testing of normality of the transformed data.

In the case when the distribution of the transformed data is normal, they are used at the application of the $\bar{x}$ and s control chart, otherwise another method must be applied for the control chart application.

3 Problem solution

The proposed method of application of the $\bar{x}$ and s chart using the Box-Cox transformation is based on a computer program, which has been developed in Excel 2013 and Visual Basic for Applications for Excel 2013.

Each stage of the proposed method is performed on a separate worksheet.

The paper presents the operation of the program by means of an example.

The input data for the presented example were generated by computer according to a Weibull distribution, with the shape parameter $\beta=2.2$, scale parameter $\eta=2.5$ and threshold parameter $\gamma=150$.

The first worksheet of the application, which is labelled “Data”, contains a table that is

used by the program user for recording the measurements (Figure 1).

Each sample may contain up to 10 measurements. In the presented example, the size of the sample is equal to 5.

	A	B	C	D	E	F	G	H	I	J
1	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
2	151.192	152.566	152.455	151.103	151.017					
3	155.37	153.32	151.722	150.577	150.543					
4	151.697	151.784	151.936	152.59	151.481					
5	151.112	153.757	152.842	153.689	152.47					

24	150.934	152.005	152.033	151.485	154.713					
25	152.688	151.573	151.478	154.037	153.359					
26	152.535	151.921	151.163	151.969	151.141					

Fig. 1. The Input data area.

The second worksheet of the program, which is labelled “AD”, is used by the program for applying the Anderson-Darling normality test for the process data according to the theoretical notions presented in [1]. The values for the number of measurements N , the average $\bar{x}$ and the standard deviation s are calculated for the measurements recorded in the worksheet labelled “Data” (Figure 2).

=COUNT(Data!A:J)		=AVERAGE(Data!A:J)				
	A	B	C	D	E	F
1	N	125		A ²	0.83602707	
2	xbar	152.18		A ⁻²	0.84116362	
3	s	1.0776		P-value	0.03033835	
4						
5	Test of normality for Process data			Decision		
6				The distribution is not normal		
=STDEV.S(Data!A:J)						

Fig. 2. The Anderson-Darling test for the process data.

The user of the program must click the control button labelled “*Test of normality for Process data*”. As a result of this action, the procedure $A_D_x()$ will be run. The listing for the first part of this procedure is presented by Figure 3.

```
Sub A_D_x()  
  Dim N As Integer, I As Integer  
  Dim xbar As Double, s As Double  
  Dim A2 As Double, A2M As Double, P As Double  
  Dim x() As Double, F() As Double  
  N = Range("B1").Value  
  xbar = Range("B2").Value  
  s = Range("B3").Value  
  For I = 1 To N  
    ReDim Preserve x(I)  
    x(I) = Application.Small(Sheets(1).Range("$A:$J"), I)  
    ReDim Preserve F(I)  
    F(I) = Application.Norm_Dist(x(I), xbar, s, 1)  
  Next I
```

Fig. 3. The procedure $A_D_x()$ – Part 1.

After the declaration of the variables, the procedure collects from the worksheet the

values for the number of measurements N , the average $\bar{x}$ and the standard deviation s . By means of a *FOR* structure, the measurements are ordered and for each of them, the cumulative distribution function is calculated. The measurements and the corresponding values of the cumulative distribution function are recorded into two arrays.

The listing for the second part of the procedure $A_D_x()$ is presented in Figure 4.

```
A2 = 0
For I = 1 To N
    A2 = A2 + (2 * I - 1) * (Application.Ln(F(I)) + Application.Ln(1 - F(N + 1 - I)))
Next I
A2 = -A2 / N - N
A2M = A2 * (1 + 0.75 / N + 2.25 / N ^ 2)
If A2M < 0.2 Then
    P = 1 - Exp(-13.436 + 101.14 * A2M - 223.73 * A2M ^ 2)
Elseif A2M >= 0.2 And A2M < 0.34 Then
    P = 1 - Exp(-8.318 + 42.796 * A2M - 59.938 * A2M ^ 2)
Elseif A2M >= 0.34 And A2M < 0.6 Then
    P = Exp(0.9177 - 4.279 * A2M - 1.38 * A2M ^ 2)
Elseif A2M >= 0.6 And A2M < 13 Then
    P = Exp(1.2937 - 5.709 * A2M + 0.0186 * A2M ^ 2)
End If
Range("E1").Value = A2
Range("E2").Value = A2M
Range("E3").Value = P
If P >= 0.05 Then
    Range("D6").Value = "The distribution is normal"
Else
    Range("D6").Value = "The distribution is not normal"
End If
End Sub
```

Fig. 4. The procedure $A_D_x()$ – Part 2.

First, it is calculated the value of A^2 , which is used for computing the value of A^{*2} , denoted into the procedure by $A2M$. The next stage is the calculation of the P -value, which depends on the value of A^{*2} . Using the P -value, the procedure makes the decision about the normality of the process data distribution.

The third worksheet of the program, which is labelled “BoxCox”, is used by the program for transforming the process data. The first columns, from A to J , are used for collecting the measurements from the worksheet “Data”. By means of the formula (2), there are calculated the standardized values of the transformed data z_i , for different values of the λ parameter.

For each sample, the value of the standard deviation, s_z is computed (Figure 5).

Fig. 5. The worksheet BoxCox – Part 1.

The standard deviation σ_z for the distribution of the variable z_i is calculated according to formula (3), by means of the values of $\bar{s}_z$ and c_4 (Figure 6).

Fig. 6. The worksheet BoxCox – Part 2.

The value of the λ parameter is calculated by means of the procedure *Calculate_Lambda()*. The listing of this procedure is presented by Figure 7.

```
Sub Calculate_Lambda()  
    Dim Lambda As Double, temp As Double  
    Dim sigma_min As Double  
    Lambda = -5  
    Range("X5").Value = Lambda  
    sigma_min = Range("W5").Value  
    temp = -4.99  
    Do While temp <= 5  
        Range("X5").Value = temp  
        If Range("W5").Value < sigma_min Then  
            sigma_min = Range("W5").Value  
            Lambda = temp  
        End If  
        temp = temp + 0.01  
    Loop  
    Range("W9").Value = sigma_min  
    Range("X9").Value = Lambda  
End Sub
```

Fig. 7. The procedure Calculate_Lambda().

This procedure will repeat all the calculations on the worksheet “BoxCox” for the λ parameter taking values into the interval $[-5, 5]$ with a step value of 0.01. In this manner, the value of λ that correspond to the minimal value of σ_z will be found. By means of this value, the transformed data are calculated using the formula (1) (Figure 8).

Fig. 8. The transformed data.

The distribution of transformed data will not always result as being normal. Therefore, the next stage that must be performed, consists of application of normality test for the transformed data. This operation will be achieved in the same manner as the test of normality for the process data (Figure 9).

	A	B	C	D	E	F
1	N	125		A ²	0.73417296	
2	ybar	1.22613E-11		A ²	0.73868372	
3	s	4.30917E-13		P-value	0.05429646	
4						
5	Test of normality for Transformed data			Decision		
6				The distribution is normal		

Fig. 9. The test of normality for the transformed data.

For the presented example, the distribution of the transformed data has resulted as being normal.

Therefore, in this case, the classic procedure of application of the *xbar* and *s* chart can be applied for the transformed data. This operation is performed on the worksheet labelled “Chart” (Figure 10).

=AVERAGE(B2:K2)			=AVERAGE(\$L\$2:\$L\$26)			=\$N2+\$U\$2*\$O2			=\$V\$2*\$O2			
	L	M	N	O	P	Q	R	S	T	U	V	W
1	ybar _i	s _i	ydblbar	sbar	UCLybar	LCLybar	UCLs	LCLs	n	A ₃	B ₄	B ₃
2	1.25E-11	3.16E-13	1.23E-11	4.08E-13	1.28E-11	1.17E-11	8.53E-13	0	5	1.427	2.089	0
3	1.22E-11	8.06E-13	1.23E-11	4.08E-13	1.28E-11	1.17E-11	8.53E-13	0				
4	1.24E-11	1.7E-13	1.23E-11	4.08E-13	1.28E-11	1.17E-11	8.53E-13	0				
.....												
25	1.21E-11	4.4E-13	1.23E-11	4.08E-13	1.28E-11	1.17E-11	8.53E-13	0				
26	1.24E-11	2.43E-13	1.23E-11	4.08E-13	1.28E-11	1.17E-11	8.53E-13	0				
=STDEV.S(B2:K2)			=AVERAGE(\$M\$2:\$M\$26)			=\$N2-\$U\$2*\$O2			=\$W\$2*\$O2			

Fig. 10. The calculations that are needed for the representation of the control chart.

The columns from *B* to *K* contain the transformed data, which are used for calculating the points of the chart, as well as the central lines and the control limits. The control chart is presented by Figure 11.

Fig. 11. The xbar and s chart for the transformed data.

It can be observed that for the presented example the process is under control.
The process capability analysis is presented by Figure 12.

Fig. 12. The process capability analysis.

The program uses the value of the λ parameter and the specification limits for calculating the transformed specification limits: LSL_{TR} and USL_{TR} . For the presented example, the value of the λ parameter is less than 0, therefore the value of LSL_{TR} is greater than the value of USL_{TR} .

In the next stage, the program collects from the worksheet labelled “Chart” the values of the center lines: $ydblbar$ and $sbar$, as well as the size of the sample, n . The value of c_4 coefficient is calculated according to the formula (4).

After the calculation of the standard deviation, σ the program computes the values of the capability indices for the transformed data: Cp_{TR} , CPL_{TR} , CPU_{TR} and CPK_{TR} .

The program also calculates the proportions of nonconforming items that correspond to the transformed data: $p_{L TR}$, $p_{U TR}$ and $p_{tot TR}$.

4 Conclusions

The presented method can be used in plant practice for controlling the processes in the case when the process data distribution is not normal, but can be converted into a normal distribution by means of the Box-Cox transformation.

The method is applied by means of a computer program which was developed so that the users do not have to possess computer programming knowledge.

Each stage of application of the proposed method is implemented on a separate worksheet.

The user of the program must record the measurements of the samples and all the calculations are performed by the program.

In order to assess the normality of process data distribution, the program uses a procedure developed on the base of the Anderson-Darling test. If the measurements distribution is normal, they can be used at the control chart application, otherwise the data will be transformed by means of a procedure developed on the base of the Box-Cox transformation.

The Anderson-Darling normality test is applied again, this time for the transformed data. In the case when the distribution of the transformed data is normal, they are used at the application of the $\bar{x}$ and s control chart, otherwise another method must be applied for the control chart application.

References

1. A.D. Sleeper, *Design for six sigma statistics* (The McGraw-Hill Companies, Inc., New York, 2006)
2. A. Mitra, *Fundamentals of quality control and improvement, third edition* (John Wiley & Sons, Inc, New Jersey, 2008)
3. G.E.P. Box, D.R. Cox, J ROY STAT SOC B MET, **26**, 211 (1964)