The research of vibrations of the beam with the attached system of mass-spring-damper using functions Tymoshenko

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Abstract. In the article [1], is considered a classic method for analyze the dynamics of the loaded beam, as a result leading to the solution of a system of transcendental equations for the eigenvalues. In this paper, shall show that perhaps the solution using functions Tymoshenko arising in the analysis of the free beam vibrations. As a result, the problem will be reduced to infinite dimensional system of differential equations satisfying the conditions for the approximate solution.

1 Introduction

The study of the beam vibrations continues to be actual over the years and finds wide application in engineering practice. The authors [4-6] studied this problem is performed analysis to find eigenfrequencies and eigenmodes. This approach is useful when considering the free vibrations of the beam. If mass or spring-mass attach to a beam, having difficulty with the calculation of eigenfrequencies. In this paper, we will show how, without changing the system of eigenfunctions, explore the fluctuations in the beam with a system of mass-spring-damper.

2 Equations of motion

Consider the free vibrations of the simply-supported uniform beam (length l, linear density ρ, stiffness EI and area of cross section F) with system of mass-spring-damper on на interior point x_0 (Fig. 1.). The stiffness of the spring g, a viscosity damper c and mass weight M. For this system is well-known equation of motion [2]:

$$EI \frac{\partial^4 U}{\partial x^4} = -\rho F \frac{\partial^2 U}{\partial t^2} \tag{1}$$

where $U(x,t)$ - function of a displacement perpendicular to the axis of the beam.
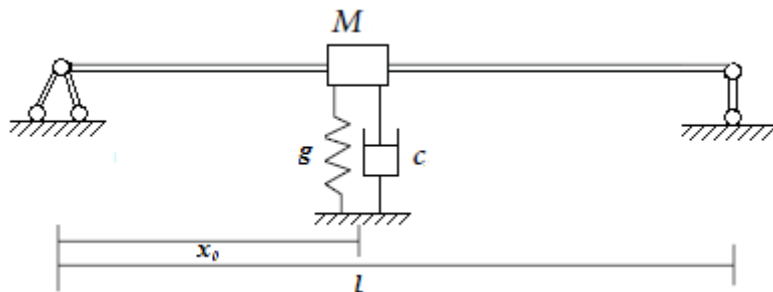
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Fig. 1. Beam with system of mass-spring-damper.

The boundary conditions are of the form

$$
\begin{cases} U\big|_{x=0} = 0 \\ \dfrac{\partial^2 U}{\partial x^2}\bigg|_{x=0} = 0 \end{cases}
\qquad
\begin{cases} U\big|_{x=l} = 0 \\ \dfrac{\partial^2 U}{\partial x^2}\bigg|_{x=l} = 0 \end{cases}
\tag{2}
$$

Influence of a system of mass-spring-damper is determined by the coupling conditions at the point x_0:

1) Continuity condition: $U\big|_{x=x_0-0} = U\big|_{x=x_0+0}$;

2) Conditions smoothness: $\dfrac{\partial U}{\partial x}\bigg|_{x=x_0-0} = \dfrac{\partial U}{\partial x}\bigg|_{x=x_0+0}$;

3) Equilibrium of moments: $EI\dfrac{\partial^2 U}{\partial x^2}\bigg|_{x=x_0-0} = EI\dfrac{\partial^2 U}{\partial x^2}\bigg|_{x=x_0+0}$;   (3)

4) Equilibrium of forces:

$$
EI\left(\frac{\partial^3 U}{\partial x^3}\bigg|_{x=x_0-0} - \frac{\partial^3 U}{\partial x^3}\bigg|_{x=x_0+0}\right) = M\frac{\partial^2 U}{\partial t^2}\bigg|_{x=x_0} + c\frac{\partial U}{\partial t}\bigg|_{x=x_0} + g\,U\big|_{x=x_0}.
$$

3 The solution using functions Tymoshenko

Apply the method of separating variables, i.e. $U(x,t) = X(x)\cdot T(t)$.

If consider only (1) and (2), the eigenfunctions would constitute functions Tymoshenko:

$$
X_n(x) = \cosh k_n x - \cos k_n x - \frac{\cosh k_n l - \cos k_n l}{\sinh k_n l - \sin k_n l}\left(\sinh k_n x - \sin k_n x\right), \text{ where } k_n = \frac{\pi n}{l}.
$$

The orthogonality condition

$$
\int_0^l X_n(x)X_n(x)\,dx = \delta_{nm},
$$

where δ_{nm} - Kronecker delta function.

Let us assume that the solution of the current problem can be represented in the form:

$$U(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n(x) \tag{4}$$

And also rightly:

$$\frac{\partial^4 U}{\partial x^4} = \sum_{n=1}^{\infty} b_n(t) X_n(x) \tag{5}$$

To reduce the size of the calculations, we agree that strokes or numbers in brackets at the upper right index indicates the order derivatives of x, and the points on functions derivative of t.

Establish the relation between $\{a_n(t)\}_{n=1}^{\infty}$ и $\{b_n(t)\}_{n=1}^{\infty}$. For this use the representation $a_n(t)$, as the Fourier coefficients and 4 times apply integration by parts and taking into account $X_n^{(IV)}(x) - k_n^4 \cdot X(x) = 0$:

$$a_n(t) = \int_0^l U(x,t) X_n(x) dx = \frac{1}{k_n^4} \int_0^l U(x,t) X_n^{(IV)}(x) dx = \frac{1}{k_n^4} \Big(U(x,t) X_n'''(x)\big|_0^l -$$

$$- \int_0^l U'(x,t) X_n'''(x) dx \Big) = \frac{1}{k_n^4} \Big(U(x,t) X_n'''(x)\big|_0^l - U'(x,t) X_n''(x)\big|_0^l + U''(x,t) X_n'(x)\big|_0^l -$$

$$- \int_0^l U'''(x,t) X_n'(x) dx \Big) = \frac{1}{k_n^4} \Big(U(x,t) X_n'''(x)\big|_0^l - U'(x,t) X_n''(x)\big|_0^l + U''(x,t) X_n'(x)\big|_0^l -$$

$$- \Big(\int_0^c U'''(x,t) X_n'(x) dx + \int_c^l U'''(x,t) X_n'(x) dx \Big) \Big) = \frac{1}{k_n^4} \Big(U(x,t) X_n'''(x)\big|_0^l - U'(x,t) X_n''(x)\big|_0^l +$$

$$+ U''(x,t) X_n'(x)\big|_0^l - U'''(x,t) X_n(x)\big|_0^l + U'''(x,t) X_n(x)\big|_{x_0-0}^{x_0+0} + \int_0^l U^{(IV)}(x,t) X_n(x) dx \Big).$$

In accordance with the boundary conditions, the first four expressions in brackets are zero. We obtain

$$a_n(t) = \frac{1}{k_n^4} \Big(\big(U'''(x_0+0,t) - U'''(x_0-0,t) \big) X_n(x_0) + \int_0^l U^{(IV)}(x,t) X_n(x) dx \Big).$$

As can be seen, the integral in the brackets is the coefficient $b_n(t)$.

From (4) follows that

$$\dot{U}(x,t) = \sum_{n=1}^{\infty} \dot{a}_n(t) X_n(x) \qquad \ddot{U}(x,t) = \sum_{n=1}^{\infty} \ddot{a}_n(t) X_n(x)$$

Substitute the expression for $a_n(t)$, have

$$a_n(t) = \frac{1}{k_n^4}\left(-\frac{1}{EI}\left(M\frac{\partial^2 U}{\partial t^2}\bigg|_{x=x_0} + c\frac{\partial U}{\partial t}\bigg|_{x=x_0} + gU|_{x=x_0}\right)X_n(x_0) + b_n(t)\right).$$

Apply (4) and obtain the final formula

$$a_n(t) = \frac{1}{k_n^4}\left(-\frac{X_n(x_0)}{EI}\sum_{s=1}^{\infty}\left(M\ddot{a}_s(t) + c\dot{a}_s(t) + ga_s(t)\right)X_s(x_0) + b_n(t)\right)$$

or

$$b_n(t) = k_n^4 a_n(t) + \frac{X_n(x_0)}{EI}\sum_{s=1}^{\infty}\left(M\ddot{a}_s(t) + c\dot{a}_s(t) + ga_s(t)\right)X_s(x_0). \qquad (6)$$

The formula is clearly seen the main difference of this method from the classic separation of variables - additional load is taken into account in the Fourier coefficients, rather than in orthogonal conditions eigenfunctions. Also note that when $M = k = c = 0$, obtain $b_n(t) = k_n^4 a_n(t)$, which is consistent with the results for the free vibration.

Write the equation (1) substituting (4) and (5)

$$EI\sum_{n=1}^{\infty}b_n(t)X_n(x) = -\rho F\sum_{n=1}^{\infty}\ddot{a}(t)X_n(x)$$

$$\sum_{n=1}^{\infty}\left(EI\cdot b_n(t) + \rho F\cdot\ddot{a}_n(t)\right)X_n(x) = 0.$$

$$EI\cdot b_n(t) + \rho F\cdot\ddot{a}_n(t) = 0, \text{ где } n = \overline{1,\infty}.$$

Using resulting formula (6) in the last equation

$$k_n^4 EIa_n(t) + X_n(x_0)\sum_{s=1}^{\infty}\left(M\ddot{a}_s(t) + c\dot{a}_s(t) + ga_s(t)\right)X_s(x_0) + \rho F\cdot\ddot{a}_n(t) = 0, \text{ где } n = \overline{1,\infty}.$$

or

$$\left(MX_n^2(x_0) + \rho F\right)\ddot{a}_n(t) + cX_n^2(x_0)\dot{a}_n(t) + \left(k_n^4 EI + gX_n^2(x_0)\right)a_n(t) =$$

$$= -X_n(x_0)\sum_{\substack{s=1\\s\neq n}}^{\infty}\left(M\ddot{a}_s(t) + k\dot{a}_s(t) + ca_s(t)\right)X_s(x_0), \text{ где } n = \overline{1,\infty}.$$

For practical application, the system can be approximately solved using numerical methods and reduction method [3], because $\left\{X_n^2(x_0)\right\}_{n=1}^{\infty}$ limited and $\lim_{n\to\infty}a_n(t) = 0$.

4 Conclusion

In this article, we have been shown a method which does not require calculation of eigenfunctions and eigenfrequencies, and reduces the problem to the infinite dimensional

system of differential equations. This system can easily be solved using numerical methods and method of reduction, since It satisfies the necessary conditions.

Since the solution uses functions Tymoshenko system it makes it easier the problem of convergence of Fourier series, and also allows a comparative analysis with non-loaded beam.

It is worth noting that all of the solution was performed for free-supported beam, but it is similar to any combination of the classic fixings. Enough to take corresponding functions Tymoshenko. This solution can be generalized to multiple systems of mass-spring-damper.

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