Efficient asymptotic basis to reduce the forced dynamic problem of viscoelastic sandwich plates

F. Boumediene¹, E.M. Daya^{2,3}, J.M. Cadou⁴, L. Duigou⁴

¹ Faculté de Génie Mécanique & Génie des Procédés, USTHB, BP 32, El Alia, 16111 Bab Ezzouar, Alger, Algeria,

² Lab d'Etude des Microstructures et Mécanique des Matériaux, UMR CNRS 7239, Université de Lorraine, Metz, France

³ Lab of Excellence on Design of Alloy Metals for low-mAss Structures 'DAMAS', University of Lorraine, Metz, France.

⁴ Institut de Recherche Dupuy de Lôme, FRE CNRS 3744, IRDL, F-56100 Lorient, France

Abstract. In this paper, an efficient asymptotic basis is proposed to reduce the forced dynamic problem of viscoelastic sandwich plates. The numerical resolution is based on the asymptotic numerical method (ANM) and finite element method (FEM). Numerical tests have been performed in the case of sandwich plates with Young's modulus variable with respect to the frequency. The comparison of the results obtained in the reduced order model with those given in the full order model shows both a good agreement and a significant reduction in computational cost.

1 Introduction

Viscoelastic sandwich materials are often used as a passive solution to reduce vibrations and noise in many domains. To increase the damping in structures, the viscoelastic material is often sandwiched between identical elastic layers. The dynamic analysis of this type of structures subjected to a harmonic load excitation $\mathbf{F}$, using the finite element method, can be written in the following form [1]:

$$[\mathbf{K}_0 + E(\omega)\mathbf{K}_v - \omega^2\mathbf{M}]\mathbf{U} = \mathbf{F} \quad (1)$$

where ω is the vibration circular frequency, $\mathbf{U}$ is the complex displacement vector of dimension $[ND]$; $\mathbf{K}_0$ and $\mathbf{K}_v$ are real constant stiffness matrices of dimension $[ND \times ND]$, $E(\omega)$ is the complex Young's modulus of the viscoelastic core, $\mathbf{M}$ is the global mass matrix of dimension $[ND \times ND]$ and $\mathbf{F}$ is the vector of the excitation amplitude of dimension $[ND]$.

Resolution of this type of problem has remained relevant to researchers until now, because of its difficulty. The problem is that all resolution methods need triangulation of Jacobean matrix which is of complex type. In addition, if the displacement vector is written with separating the real and imaginary parts as follows: $\mathbf{U} = \{\mathbf{U}^R, \mathbf{U}^I\}$; the equation (1) will have two time the number of degrees of freedom (*dof*) of equivalent real

problem. Then resolution could lead to high computational cost in the case of large scale structures.

Recently, we have proposed a new basis to reduce this type of problem in the cases of beams [2] and plates with constant Young's modulus [3]. In this paper, this reduction basis is improved to be applied in the case of sandwich structure with Young's modulus dependent on the frequency. For the numerical computation, structures are discretized using shell element with eight nodes and eight degrees of freedom by node [5]. The reduction method is combined to the asymptotic numerical method (ANM) [4] to compute the response of viscoelastic sandwich plate.

2 Asymptotic numerical method (ANM)

The dynamic governing equation can be written in the following form:

$$\mathbf{R}(\mathbf{X}) = \mathbf{0} \quad (2)$$

where $\mathbf{R}$ is the residual vector of dimension $[ND]$, the vector $\mathbf{X} = (\mathbf{U}, \omega, E)$ contains all the unknowns of the problem.

The resolution of equation (2), using ANM, are widely explained in different papers [2,4]. Here we explain only the principle.

Within the ANM we can solve the residual equation (2) in a step by step manner, where the unknown

$\mathbf{X}$ is sought in the form of truncated power series with respect to a path parameter “ a ” [2]:

$$\mathbf{X}(a) = \mathbf{X}_0 + \sum_{p=1}^n a^p \mathbf{X}_p \quad (3)$$

where $\mathbf{X}_0$ is a regular initial solution of equation (2), $\mathbf{X}_p$ is the unknown, at each order p ($p = 1, n$) to be computed and n is the truncation order.

Series (3) are inserted in equations (2). Then, from the identifications of the like powers of ‘ a ’, a set of recurrent linear problems is obtained at each order p as follows:

$$\mathbf{K}_t \mathbf{X}_p = \mathbf{F}_p^{nl} \quad (4)$$

where $\mathbf{K}_t$ denotes the Jacobian matrix at the initial point $\mathbf{X}_0$ and $\mathbf{F}_p^{nl}$ is the second member vector at order p . The second member is known since it depends only on known parameters $\mathbf{X}_i$ computed in precedent orders ($i = 1, p-1$).

The limit of validity of series (3) can be computed automatically using the tolerance ε . Then, resolution of one step needs one matrix triangulation and resolution of p linear systems (4). This leads to a significant computational time. For that, we propose to use reduction model.

3 Reduction model

To build the projection matrix, one ANM step without reduction is realized and vectors at each order $\mathbf{U}_i$ issued from series (3) are saved to build the basis $\mathbf{R}$. After that, the computation is carried out in a reduced form. The reduction consists of projecting the unknown vector onto a small basis as follows:

$$\mathbf{U} = \mathbf{R} \cdot \mathbf{u} \quad (5)$$

where $\mathbf{R}$ is the projection matrix of dimension $[ND \times nd]$ and $\mathbf{u}$ is the reduced vector of dimension $[nd]$.

Two reduction algorithms are tested:

- In the first one (noted RED1), the reduction is applied directly to equation (1): After replacing the vector $\mathbf{U}$, by its expression (5), and multiplying the left side by $\mathbf{R}^T$; the governing equation to resolve becomes smaller. Its dimension is now $[nd]$ with ($nd \ll ND$) :

$$[\mathbf{k}_0 + E(\omega)\mathbf{k}_v - \omega^2\mathbf{m}]\mathbf{u} = \mathbf{f} \quad (6)$$

where $\mathbf{k}_0 = \mathbf{R}^T \mathbf{K}_0 \mathbf{R}$, $\mathbf{k}_v = \mathbf{R}^T \mathbf{K}_v \mathbf{R}$, $\mathbf{m} = \mathbf{R}^T \mathbf{M} \mathbf{R}$ and $\mathbf{f} = \mathbf{R}^T \mathbf{F}$

Then, the solution of the reduced problem (6) is performed by the ANM but only reduced matrices and vectors are considered (Figure 2). This procedure works well in the case of beam [2].

- In the second one (noted RED2), the reduction is applied to resolve only equation (4) in the application of the ANM to the problem. Then only matrix decomposition is done in the reduced form. The second member is computed in a complete order (Figure 2).

After that, we reduce it to resolve the problem (4) in a reduced forme, as follow:

$$\mathbf{k}_t \mathbf{x}_p = \mathbf{f}_p^{nl} \quad (7)$$

where $\mathbf{k}_t = \mathbf{R}^T \mathbf{K}_t \mathbf{R}$ and $\mathbf{f}_p^{nl} = \mathbf{R}^T \mathbf{F}_p^{nl}$

This procedure was applied to a viscoelastic material with a constant Young's modulus [5].

Fig. 1 Resolution chart

4 Numerical examples

The first plate (plate 1) has two clamped edges with dimensions $(348 \times 304.8 \times 3.175 \text{ mm}^3)$ (Figure 2). Its characteristics are given in Table 1. The Young's modulus is supposed variable with respect to the frequency following the generalized Maxwell model:

$$E(\omega) = k_0 + \eta_0 \cdot i\omega + \sum_{j=1}^{N_{max}} i\omega / \left(\frac{i\omega}{k_j} + \frac{1}{\eta_j} \right) \quad (8)$$

where k_j and η_j are material Maxwell constants (given experimentally) and ‘ i ’ is the complex number ($i^2 = -1$).

The plate is discretized into 173 nodes ($ND = 2768$ d.o.f, real and imaginary parts). The transverse excitation force ($F=100\text{N}$) is applied to the center of the plate.

Fig. 2. Clamped-free-clamped-free plate (Plate 1)

Table 1. Material properties

Material properties	Elastic layers	Viscoelastic layer
Young's modulus $[\text{N/m}^2]$	$0,21 \cdot 10^{12}$	$0,2722 \cdot 10^7$

Poisson's ratio	0,3	0,44
Density [kg/m ³]	7800	1200
Thickness [mm]	1,524	0,127

Results from the full order model (using ANM without reduction) and those from the reduced order model with basis containing twenty vectors ($nd=40$ real and imaginary parts), are superposed for comparison (Figure 3). To confirm the validity of our program, results are compared with those computed directly by inverting the tangent matrix at each frequency as follows:

$$\mathbf{U}(\omega) = [\mathbf{K}_0 + E(\omega)\mathbf{K}_v - \omega^2\mathbf{M}]^{-1} \cdot \mathbf{F} \quad (9)$$

Figure 3, represents the magnitude of the transverse displacement evolution with respect to frequency at the load position. Figure 4, presents the residual value evolution in a logarithm scale where the residual value is computed as follows:

$$RES = \|\mathbf{R}(\mathbf{X})\| / \|\mathbf{K}_0 \cdot \mathbf{U}\| \quad (10)$$

Fig. 3. Transverse displacement magnitude (Plate 1)

Fig. 4. Residual value evolution (Plate 1)

These figures show good agreement between results with the 2nd strategy (RED2) and ANM results around the first mode until a frequency equal to 50Hz. However, the first strategy (RED1) gives a large error. From study of the computational time, we found that the matrix triangulation consumes more time in one step. For example, the resolution step time is equal to (4.84E-02) the decomposition time for a problem with (2768 dof) and 0.004 the decomposition time for a problem with (9104 dof)

Then avoiding triangulation of full order matrices is sufficient to reduce significantly the computational time. Therefore, we prefer keeping the second strategy which

consists of reducing only the linear equations at each order p . But we also remark, that this strategy needs a basis updating.

To control the solution accuracy, the basis is updated if the residual value is greater than the tolerance (1.E-4), we proceed as follows:

1. recalculate a new exact solution (9)
2. do one ANM step in a full order model,
3. update the basis.
4. resolution in reduced form and verify residual value. If it is greater than the tolerance return to 1.

Following this procedure, we found good result (Figure 5) but five update steps are needed (Figure 6). Then, this requires five computations in full size model. So to decrease this number, it is possible to correct the solution at each order to avoid error accumulation. This is the objective of the next section.

Fig. 5. Transverse displacement with basis update (Plate 1)

Fig. 6. Residual value evolution with basis update (Plate 1)

6 Reduction-correction techniques

In this section, we present the method used to improve the validity of the reduced solution. We consider the reduced solution, $\mathbf{X}_p^{\text{red}}$, as already known at the order of truncation p . This vector is issued from resolution of the problem (7). The solution of the full-size problem at the order p can be defined by:

$$\mathbf{X}_p = \mathbf{X}_p^{\text{red}} + \Delta\mathbf{X}_p \quad (11)$$

where $\Delta\mathbf{X}_p$ is the error.

We replace this expression in equation (4):

$$\mathbf{K}_t(\mathbf{X}_p^{\text{red}} + \Delta\mathbf{X}_p) = \mathbf{F}^{\text{nd}} \quad (12)$$

Then, the error $\Delta\mathbf{X}_p$ can be computed as follows (13):

$$\mathbf{K}_t(\Delta \mathbf{X}_p) = \mathbf{F}^{nd} - \mathbf{K}_t(\mathbf{X}_p^{red}) \quad (13)$$

Instead of taking the true matrix $\mathbf{K}_t$, we use the factorized matrix used in the first full-size computation where the basis has been defined. This procedure can be repeated until an accepted residual value is obtained. With this procedure, we success to find all the solution curves of figures 5 and 6 with only three basis updates (Figure 7).

Fig. 7. Residual value evolution with correction step and basis update (Plate 1)

The method is now applied to simply supported plate (figure 8). Geometrical and material characteristics are the same as the plate 1. We remark the good agreement between the Full order model (FOM) and the reduced order model (ROM) results (Figure 9) with only three basis updates (Figure 10).

Fig. 8. Plate 2

Fig. 9. Transverse displacement with correction step and basis update (Plate 2)

Fig. 10. Residual value evolution with correction step and basis update (plate 2)

8 Conclusion

Many tests have been performed on plates with different boundary conditions and viscoelastic properties. All examples treated confirmed that the proposed reduced basis is efficient and reproduces perfectly the results issued from the full order model. In addition, a significant computational gain is obtained by using the reduction-correction procedure, up to 80% with considering the time needed to build the basis and to compute the reduced initial vectors and matrices.

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