MATEC Web of Conferences **83**, 01001 (2016) DOI: 10.1051/matecconf/20168301001
CSNDD 2016

The role of parameters of smallness in deduction of approximated theories for deterministic dynamics of beams

Enrico Babilio[1,a] and Stefano Lenci[2,b]

[1] *Department of Structures for Engineering and Architecture (DiSt), University of Naples "Federico II", 80134 Naples, Italy*
[2] *Department of Civil and Building Engineering and Architecture (DICEA), Polytechnic University of Marche, 60131 Ancona, Italy*

Abstract. This paper considers the role played by the order of magnitude of terms involved in the equations of motion of a geometrically exact beam theory. Upon introducing a smallness parameter, a procedure for derivation of approximated models is sketched.

1 Introduction

In many real-world applications, technical rules and codes enforce structures to fulfill specific limitations. For instance, often structures of engineering interest are not allowed to exhibit displacements, rotations or deformations larger than assigned values. Thus, accordingly, their behavior can be investigated by using approximated models able to furnish information which is completely satisfactory from a practical point of view, at reduced computational cost in comparison with more refined models. The present short contribution is focused on the derivation of approximated beam models from a parent geometrically exact nonlinear Timoshenko theory, depending on axial and transversal displacements u, v and on a shear angle γ. The orders of magnitude of such functions, or their derivatives involved in the equations of motions, are relevant to understanding which term is negligible and which other should be retained in deducing approximated models. To this end, a parameter of smallness ϵ is introduced and u, v, γ, and their derivatives, are replaced by suitably rescaled functions of order of unity times powers of ϵ. By taking into account terms of same power of ϵ, a hierarchy of models can be derived.

2 The geometrically exact model

Let us consider an initially straight beam undergoing planar and twist-less deformed states and further assume that cross sections remains flat and undistorted, but not necessarily orthogonal to the beam axis during the deformation process. A small element of initial length dx, in reference and deformed configuration, is depicted in figure 1, to which we refer for notations: $u(x,t)$ and $v(x,t)$ stand for the axial and transversal displacements of the beam axis,

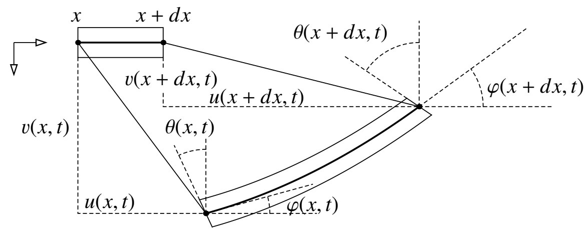

Figure 1. Element in reference and deformed configurations

$\varphi(x,t)$ and $\theta(x,t)$ are the beam axis rotation and the cross-sectional rotation. We assume that displacements (and forces) are positive if concordant with the reference (see vectors in figure 1) and rotations (and moments) are positive if counterclockwise oriented.

The length of the deformed axis of the beam element is given by

$$\mathrm{d}s = \sqrt{2\,\varepsilon + 1}\,\mathrm{d}x\,, \tag{1}$$

where ε is the axial strain written as

$$\varepsilon = u' + \frac{u'^2}{2} + \frac{v'^2}{2}\,, \tag{2}$$

the prime denoting derivative with respect to x.

The rotation φ is related to displacements u and v by

$$\varphi = -\arctan\left(\frac{v'}{1+u'}\right). \tag{3}$$

In order to measure the shear deformation, the angle $\gamma(x,t)$ is further introduced and its relationship with φ and θ is stated as

$$\gamma = \theta - \varphi\,. \tag{4}$$

[a] e-mail: enrico.babilio@unina.it
[b] e-mail: lenci@univpm.it

Let us finally introduce the function

$$\kappa = \gamma' + \frac{v'u'' - \left(1+u'\right)v''}{\left(1+u'\right)^2 + v'^2} , \qquad (5)$$

which is nothing but θ', i.e. the derivative of θ with respect to x, and, on adopting nomenclatures introduced in [1], [2] or [3], it can be called the normalized, flexural or mechanical curvature, respectively.

On introducing the virtual variations of Eqs. (2), (4) and (5) and the properly conjugated generalized stresses N, T and M, the equations of motion can be derived from the Principle of Virtual Work as (see [4])

$$\rho\, A\, \ddot{u} + c\, \dot{u} = \left(N\frac{\mathrm{d}s}{\mathrm{d}x}(1+u') - Tv'\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^{-1}\right)' , \qquad (6)$$

$$\rho\, A\, \ddot{v} + c\, \dot{v} = \left(N\frac{\mathrm{d}s}{\mathrm{d}x}v' + T(1+u')\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^{-1}\right)' , \qquad (7)$$

$$M' + \rho\, I\, \ddot{\theta} = T\,\sqrt{2\varepsilon + 1} , \qquad (8)$$

where A, I, ρ and c are the cross-sectional area, the second moment of area, the mass density and the damping coefficient, respectively, and the overdot denotes derivative with respect to time t.

Notice that Eqs. (6)-(8) are geometrically exact.

Finally, as constitutive law we use the standard linear elastic relationship (see e.g. [5])

$$N = E\,A\,\varepsilon\,, \quad T = G\,A\,\gamma\,, \quad M = E\,I\,\kappa\,, \qquad (9)$$

where E is the Young's modulus and G is the (second) Lamé's constant.

It is worth noting that although N, T and M by virtue Eqs. (9) are linearly dependent of the relevant kinematic variables ε, γ and κ, respectively, they however are nonlinear functions of u' and v'.

2.1 A remark on the moment-curvature relation

It is known that beam theories completely decoupled from the three-dimensional theory do not provide any structure for constitutive equations, as typically happens for models belonging to the class of Cosserat beam theories [6] (for a systematic treatment of beams as oriented bodies see also [7]). Thus, constitutive equations must be either introduced on an axiomatic way or experimentally established. For considerations on how to obtain constitutive equations, see [8].

For the sake of this communication, we axiomatically assume that Eqs. (9) are satisfactory relationship. While a deeper discussion on this aspect is beyond the scope of the present contribution (and is extensively treated in [4]), here we emphasize that, interestingly, in formulating a constitutive relationship for the bending moment, a key question treated in many papers and books is which, between the normalized curvature $\kappa = \theta'$ and the derivative of θ with respect to the arclength s, which is also a curvature, is the more suitable measure to adopt for beams undergoing large deflections and rotations.

Actually, both are used in the scientific literature. Arguments in favor of the former can be found in [1–3, 6, 9]. On the other hand, in [10–14] the latter is treated as the better choice.

3 A sketch on approximated models

The orders of magnitude of functions involved in Eqs. (2), (4) and (5) and in Eqs. (6)-(8) play relevant role in deriving approximated models of beams with different nonlinear dominant terms.

To compare orders of magnitude, let us introduce a real positive parameter ϵ much smaller than unity. Soon arises the question how small ϵ is. Indeed, the smallness of a certain function of engineering interest depends on the application under investigation and may differ from a technical problem to another.

In order to give a numerical estimation of ϵ in a practical application, let us consider a simply supported beam and assume that a single-wave deformed shape, such as

$$v(x,t) = \alpha(t)\sin\left(\frac{\pi x}{L}\right) , \qquad (10)$$

be in equilibrium with a given load. Shape (10) allows us to write

$$\sup_{x\in[0,L]} |v| = |\alpha| , \qquad \sup_{x\in[0,L]} |v'| = \frac{\pi}{L}|\alpha| . \qquad (11)$$

If prescriptions about $|\alpha|$ are somehow known, we may agree to accept

$$\epsilon = \frac{\pi}{L}|\alpha| . \qquad (12)$$

For instance, the Italian Code (NTC2008) enforces $|\alpha| \leq L/250$ for floors under exercise (dead plus live) loads in buildings. Hence, in such a specific case, ϵ can be taken as about 10^{-2}.

Now, let us assume that u, v and γ, and all their derivatives up to the requested order, may be approximated through their Taylor polynomials about a specific configuration, such as for instance the static equilibrium position or the undeformed configuration and introduce functions $\hat{u}, \hat{u}', \ldots, \hat{v}, \hat{v}', \ldots, \hat{\gamma}, \hat{\gamma}', \ldots$, of order of unity ($\epsilon^0$), which are rescaled versions of $u, u', \ldots, v, v', \ldots, \gamma, \gamma', \ldots$, satisfying the same regularity requests enforced for the original counterparts.

Notice that, with these assumptions, since the function $\hat{u}'$ is of order of ϵ^0, it differs from the space derivative $\partial\hat{u}/\partial x$. Indeed, they have the same shape, but different order of magnitude.

3.1 Shear angle limitations and consequences

Recalling that the beam dynamics depends on functions u, v and γ and their derivatives, the order of magnitude of all of them must be known to derive approximated models. We discuss about the shear angle γ, first. Indeed, we assume that a suitable γ is in the range $(-\pi/18, \pi/18)$, which should be true for most practical applications. Through the Maclaurin expansion and retaining up to linear terms,

$$\sin\gamma \simeq \gamma\,, \qquad \cos\gamma \simeq 1\,, \qquad (13)$$

where "≃" indicates approximated equalities. Notice that Eqs. (13) imply an absolute error of at most 0.5 and 1.5 percent, at $\pm\pi/18$, in estimating sine and cosine, respectively. Thus, we may accept that γ is of order $\sqrt{\epsilon}$, at most. Furthermore, we may assume that γ has a very small space variation (along the beam-axis) during the entire deformation process, implying that we retain γ' is of order ϵ.

In what follows, we prescribe the orders of magnitude for u' and v' and introduce approximations (ϵ^m, ϵ^n), being m and n the orders of ϵ chosen for u' and v', respectively.

3.2 Approximation (ϵ^1, ϵ^1)

On assuming that both u' and v' are of order ϵ, that is

$$u' = \epsilon \hat{u}' \ , \qquad v' = \epsilon \hat{v}' \ , \tag{14}$$

the axial strains ε in Eq. (2) turns out to be of order of ϵ, and, neglecting terms multiplied by powers of ϵ greater than one, is approximated as

$$\varepsilon \simeq \epsilon \hat{u}' = u' \ , \tag{15}$$

which is that usually adopted for small strain beam theory.

Beam axis rotations are of order of ϵ, too, and are approximated by

$$\varphi \simeq -\epsilon \hat{v}' \ , \tag{16}$$

as it can be seen developing Eq. (3) in Taylor series and retaining terms up to the first order of ϵ. If it is further assumed that also γ is of order ϵ, that is

$$\gamma = \epsilon \hat{\gamma} \ , \tag{17}$$

the cross-sectional rotation may be approximated as

$$\theta \simeq \epsilon(\hat{\gamma} - \hat{v}') \ . \tag{18}$$

The assumptions (14) inserted in Eq. (5), neglecting all terms whose order of magnitude is certainly smaller than ϵ, the bending curvature (5) takes the form

$$\kappa \simeq \epsilon \hat{\gamma}' - v'' + \epsilon \hat{v}' u'' \ . \tag{19}$$

Up to now, only hypotheses on smallness of u', v' and γ' have been assumed, but nothing has been said about u'' and v''. Hence, the order of magnitude of κ, which depends on the second derivatives of u and v, is still unknown.

Without specific information about u'' and v'', we may assume, at least in principle, that they are of any order. We restrict to five powers of ϵ, namely u'' and v'' are of order ϵ^1, $\epsilon^{\frac{1}{2}}$, ϵ^0 $\epsilon^{-\frac{1}{2}}$ or ϵ^{-1}, leading to the approximated expressions of curvature reported in Table 1, with the order of u'' on rows, that of v'' on columns. The order of magnitude of curvatures is also indicated in Table 1, recalling that all *hatted* functions are of order of magnitude of unity.

We observe that the curvature is of order ϵ if and only if v'' is ϵ and u'' is ϵ^0 at most. If u'' is of order of ϵ, or $\sqrt{\epsilon}$, we get back the curvature adopted in small strain theory, while if u'' is ϵ^0, curvature contains the nonlinear term $v'u''$. In all other cases reported, curvature has magnitude greater than ϵ, hence γ', which is of order of ϵ, is negligible. Table 1 shows also that curvature is nonlinear whenever u'' is greater of v'' of at least two order of ϵ.

3.3 Approximation $(\epsilon^1, \epsilon^{\frac{1}{2}})$

If u' is of order of v'^2, that is

$$u' = \epsilon \hat{u}' \ , \qquad v' = \sqrt{\epsilon} \hat{v}' \ , \tag{20}$$

the approximated ε, which is of order of ϵ, takes the form

$$\varepsilon \simeq \epsilon \left(\hat{u}' + \frac{\hat{v}'^2}{2} \right) \ , \tag{21}$$

known as von Kármán strain [15], usually adopted for beams undergoing moderately large deflections.

Rotations are of order of $\sqrt{\epsilon}$ and are written as

$$\theta \simeq -\sqrt{\epsilon} \hat{v}' \ , \quad \text{or} \quad \theta \simeq \sqrt{\epsilon}(\hat{\gamma} - \hat{v}') \ , \tag{22}$$

respectively depending on if it is assumed that γ is of order ϵ (as u') or $\sqrt{\epsilon}$ (as v'), that is

$$\gamma = \epsilon \hat{\gamma} \ , \quad \text{or} \quad \gamma = \sqrt{\epsilon} \hat{\gamma} \ . \tag{23}$$

The curvature takes the form

$$\kappa \simeq \epsilon \hat{\gamma}' - v'' + \sqrt{\epsilon} \hat{v}' u'' \ . \tag{24}$$

under the hypothesis that γ' is of order ϵ.

Table 2 reports terms which are dominant in the curvature expression. The order of magnitude is reported as well. We see that the curvature is of order ϵ if and only if v'' is ϵ and u'' is $\sqrt{\epsilon}$ at most. If u'' is of order of ϵ we get back the curvature adopted in both small strain and moderately large deflections theories, while if u'' is $\sqrt{\epsilon}$, curvature contains the nonlinear term $v'u''$. In all other cases reported, the derivative of the shear angle (recall that γ' is assumed of order ϵ) is negligible. Table 2 shows also that curvature is nonlinear whenever u'' is greater of v'' of at least one order of ϵ.

4 Summary and conclusions

The present short communication is focused on the derivation of approximated models from a geometrically exact beam theory, in which no assumptions on the order of magnitude of displacements, rotations and deformations are taken *a priori*.

Since the order of magnitude of each term allows understanding which one is negligible in approximated models, a parameter of smallness ϵ is introduced and relevant terms appearing in equations of motion are replaced by suitably rescaled functions of order of unity times powers of ϵ.

To explain the idea, axial strain, rotation and a hierarchy of approximated curvatures are derived under the hypothesis that v' and γ are of order ϵ or $\sqrt{\epsilon}$ and u' and γ' are of order ϵ.

It must be noted that to achieve the approximated versions of Eqs. (6)-(8), it is still necessary to infer order of magnitude of $\dot{u}$ and $\ddot{u}$ in Eq. (6), $\dot{v}$ and $\ddot{v}$ in Eq. (7), $\dot{u}', \ddot{u}', \dot{v}', \ddot{v}'$ in Eq. (8) (nested in $\ddot{\theta}$). The task can be performed as shown for other terms and then, by substituting

Table 1. The curvature in (ϵ^1, ϵ^1) approximation

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="5">v''</td></tr>
<tr><td>ϵ</td><td>$\sqrt{\epsilon}$</td><td>1</td><td>$\frac{1}{\sqrt{\epsilon}}$</td><td>$\frac{1}{\epsilon}$</td></tr>
<tr><td rowspan="5">u''</td><td>ϵ</td><td rowspan="2">$\epsilon(\hat{\gamma}' - \hat{v}'')$</td><td rowspan="3">$-\sqrt{\epsilon}\hat{v}''$</td><td rowspan="4">$-\hat{v}''$</td><td rowspan="5">$-\frac{\hat{v}''}{\sqrt{\epsilon}}$</td><td rowspan="5">$-\frac{\hat{v}''}{\epsilon}$</td></tr>
<tr><td>$\sqrt{\epsilon}$</td></tr>
<tr><td>1</td><td>$\epsilon(\hat{\gamma}' - \hat{v}'' + \hat{v}'\hat{u}'')$</td></tr>
<tr><td>$\frac{1}{\sqrt{\epsilon}}$</td><td>$\sqrt{\epsilon}\hat{v}'\hat{u}''$</td><td>$\sqrt{\epsilon}(-\hat{v}'' + \hat{v}'\hat{u}'')$</td></tr>
<tr><td>$\frac{1}{\epsilon}$</td><td colspan="2">$\hat{v}'\hat{u}''$</td><td>$-\hat{v}'' + \hat{v}'\hat{u}''$</td></tr>
</table>

Table 2. The curvature in $(\epsilon^1, \epsilon^{\frac{1}{2}})$ approximation

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="5">v''</td></tr>
<tr><td>ϵ</td><td>$\sqrt{\epsilon}$</td><td>1</td><td>$\frac{1}{\sqrt{\epsilon}}$</td><td>$\frac{1}{\epsilon}$</td></tr>
<tr><td rowspan="5">u''</td><td>ϵ</td><td>$\epsilon(\hat{\gamma}' - \hat{v}'')$</td><td rowspan="2">$-\sqrt{\epsilon}\hat{v}''$</td><td rowspan="3">$-\hat{v}''$</td><td rowspan="4">$-\frac{\hat{v}''}{\sqrt{\epsilon}}$</td><td rowspan="5">$\frac{-\hat{v}''}{\epsilon}$</td></tr>
<tr><td>$\sqrt{\epsilon}$</td><td>$\epsilon(\hat{\gamma}' - \hat{v}'' + \hat{v}'\hat{u}'')$</td></tr>
<tr><td>1</td><td>$\sqrt{\epsilon}\hat{v}'\hat{u}''$</td><td>$\sqrt{\epsilon}(-\hat{v}'' + \hat{v}'\hat{u}'')$</td></tr>
<tr><td>$\frac{1}{\sqrt{\epsilon}}$</td><td colspan="2">$\hat{v}'\hat{u}''$</td><td>$-\hat{v}'' + \hat{v}'\hat{u}''$</td></tr>
<tr><td>$\frac{1}{\epsilon}$</td><td colspan="3">$\frac{\hat{v}'\hat{u}''}{\sqrt{\epsilon}}$</td><td>$\frac{-\hat{v}'' + \hat{v}'\hat{u}''}{\sqrt{\epsilon}}$</td></tr>
</table>

approximated functions in Eqs. (6)-(8) and retaining terms of the same order, approximated equation are deduced.

Finally, we stress that the advantage of having a hierarchy of approximated models relies on the possibility of choosing the more suitable set of equations for a given problem, at the less computational cost for the desired level of accuracy.

References

[1] A.H. Nayfeh, P.F. Pai, *Linear and Nonlinear Structural Mechanics* (Wiley, New York, 2004)

[2] W. Lacarbonara, *Nonlinear Structural Mechanics. Theory, Dynamical Phenomena and Modeling* (Springer, New York, 2013)

[3] A. Luongo, D. Zulli, *Mathematical Models of Beams and Cables* (ISTE-Wiley, London UK, Hoboken NJ, 2013)

[4] E. Babilio, S. Lenci, submitted (2016)

[5] S. Timoshenko, 1995 *Strength of Materials* (Krieger Publishing Company, New York, 1995, third edition)

[6] S.S. Antman, *Problems of Nonlinear Elasticity* (Springer, New York, 2005)

[7] P. Villaggio, *Mathematical Models for Elastic Structures* (Cambridge, Cambridge University Press, 1997)

[8] E. Reissner, J. Appl. Math. Phys. ZAMP, **23**, 795-804 (1972)

[9] J. Gerstmayr, H. Irschik, J. Sound Vib., **318**, 461-487 (2008)

[10] D.H. Hodges, AIAA J. **22**(12), 1825-1827 (1984)

[11] M. Berzeri, A.A. Shabana, J. Sound Vib., **235**(4), 539-565 (2000)

[12] E. B. Magrab, *Vibrations of Elastic Systems With Applications to MEMS and NEMS* (Springer, New York, 2012)

[13] E. Babilio, Eur. Phys. J.-Spec. Top., **222**(7), 1519–1539 (2013)

[14] S. Lenci, F. Clementi, G. Rega, Meccanica, DOI 10.1007/s11012-016-0374-6 (2016)

[15] J.N. Reddy, P.Mahaffey, J. Coupled Syst. Multiscale Dyn., **1**(1), 120–134 (2013)