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Method of Quasi-Optimal Synthesis Using Invariants

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Abstract. This paper studies the problem of synthesis of terminal control of a dynamic system. It is shown that to solve the problem the convolution product of objective functional and Gaussian constraint can be used as expanded functional. The use of needle variation allows to get a required condition of the objective functional minimum and to reduce the optimization problem to the boundary problem, which can be solved in closed form. The quasi-optimal solution to the problem of optimal speed of operation is obtained. The modes of operation with choosing the control law parameters were studied.

1 Introduction

The research results show [1]-[5] that the convolution product of the objective functional and the action integral gives the control structure accurate to a synthesis function. The function can be built using the stationarity conditions of the energy invariants which can be used to determine the switching surface. This allows to find that solutions to the extreme problem which satisfy to variational principle (the dynamics base) and provide the stability of the controlled motion in accordance with A.M. Lyapunov's theorem.

In this work it is shown that the application of combined-maximum principle methodology for finding a required condition for the expanded functional minimum as a convolution product of the objective criterion and the Gaussian constraint [4] provides a synthesis of quasi-optimal controls. In contrast to known results this leads to a solvable boundary problem and doesn't require to build the synthesis function.

The aim of this investigation is synthesis of terminal control using criterion of operation speed using Gaussian constraint.

2 Definition of the synthesis problem

According to the Gauss principle at every time t the dynamic system moves in such a way that constraint [5]

$$Z = \sum_{s=1}^{n} \frac{1}{2} m_s \left(\ddot{q}_s - \frac{Q_s}{m_s} \right)^2, s = \overline{1,n} \tag{1}$$

corresponding to actual way is minimized by the accelerations $\ddot{q}_s$, so that

$$\delta Z = 0, \tag{2}$$

where m_s is the material point mass; q_s – is the coordinate of the material point relative to a static Cartesian coordinate system; Q_s is the resultant force applied to the material point; n is a number of degrees of freedom of the dynamic system.

From (2) the Appell equations [5] follow in the form:

$$\frac{\partial G}{\partial \ddot{q}_s} = Q_s, \tag{3}$$

where G is the Gibbs function, and the time derivative is denoted by two dots.

Let the dynamics of the studied system satisfies (1) and is described by equations (3).

It is required to find the restricted possible forces $\mathbf{Q} \in \overline{G}_Q$, which transfer system (3) from the initial state $t = t_0$ to the final state $t = t_1$, under the condition of minimum of the objective functional

$$J = \int_{t_0}^{t_1} F(\mathbf{q}) dt \to \min, \tag{4}$$

where a convex function $F(\mathbf{q})$ is of constant sign and is continuous along with its partial derivatives in the whole domain, and t_0, t_1 is the time of start and finish of the control accordingly [2, 6].

3 Method of synthesis using invariants

The searching for the required condition for the objective functional minimum (4) is performed by the Lagrange undetermined multipliers method. Let us consider the expanded functional [4]

$$J_1 = J + \int_{t_0}^{t_1} \lambda Z dt \to \mathbf{min}, \tag{5}$$

where λ – is the Lagrange undetermined multiplier.

Let the arbitrary generalized force is determined by expression:

$$\mathbf{Q} = \hat{\mathbf{Q}} + \mathbf{\delta Q}, \tag{6}$$

where $\hat{\mathbf{Q}}$ – is the generalized force minimizing the objective functional, $\delta\mathbf{Q} = 0$ when $t \overline{\in} [\tau, \tau + \Delta t]$, $\tau \in (t_0, t_1)$ is the given continuity point of the function $\hat{\mathbf{Q}}(t)$, $\Delta t \in [\tau, t_1]$ is the given small finite time interval; $\Delta t > 0$.

Then the full variation of the functional has the following form:

$$\Delta J_1 = [\lambda Z + F]\Delta t\Big|_{t_0}^{t_1} + \sum_{s=1}^{n} \int_{t_0}^{t_1} [\lambda \delta Z + \delta' F] dt = \\ = [\lambda Z + F]\Delta t\Big|_{t_0}^{t_1} + \sum_{s=1}^{n} \int_{t_0}^{t_1} \left[\lambda \left(\frac{\partial Z}{\partial \ddot{\hat{q}}_s} \delta \ddot{q}_s \right) + V_s \delta q_s \right] dt \geq 0, \tag{7}$$

where $V_s = \dfrac{\partial F}{\partial \hat{q}_s}$ is the fictious generalized force.

The relations on the trajectory ends are the transversality conditions:

$$\lambda Z + F = 0, \tag{8}$$

if the interval $[t_1 - t_0]$ is fixed, or

$$\Delta t = 0 \tag{9}$$

if the interval $[t_1 - t_0]$ is not fixed.

When $t \in [t_0, \tau]$ the varied force and the minimizing (4) force coincide, so $\Delta J_1 = 0$.

When $t \in [\tau, \tau + \Delta t]$ $\Delta J_1 \neq 0$ and

$$\delta Z_{t \in [\tau, \tau + \Delta t]} = \frac{\partial \sum_{s=1}^{n} \frac{1}{2} m_s \left(\ddot{q}_s - \frac{Q_s}{m_s} \right)^2}{\partial \ddot{\hat{q}}_s} \delta \ddot{q}_s = \\ = \sum_{s=1}^{n} (m_s \ddot{q}_s - Q_s) \delta \ddot{q}_s. \tag{10}$$

When $t \in [\tau + \Delta t, t_1]$ $\Delta J_1 \neq 0$, but the arbitrary force Q_s and the minimizing (4) force $\hat{Q}_s$ coincide, then

$$\delta Z_{t \notin [\tau, \tau + \Delta t]} = \frac{\partial \sum_{s=1}^{n} \frac{1}{2} m_s \left(\ddot{q}_s - \frac{\hat{Q}_s}{m_s} \right)^2}{\partial \ddot{\hat{q}}_s} \delta \ddot{q}_s = \\ = \sum_{s=1}^{n} \left(m_s \ddot{q}_s - \hat{Q}_s \right) \delta \ddot{q}_s. \tag{11}$$

An increment of the integrand of the objective functional F is calculated in the following way:

$$\delta' F_{t \notin [\tau, \tau + \Delta t]} = \sum_{s=1}^{n} \frac{\partial F}{\partial \hat{q}_s} \delta q_s = \sum_{s=1}^{n} \hat{V}_s \left(\hat{\mathbf{Q}} \right) \delta q_s. \tag{12}$$

Then (7) takes the form:

$$\Delta J_1 = \sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} [V_s \delta q_s + \lambda (m_s \ddot{q}_s - Q_s) \delta \ddot{q}_s] dt + \\ + \sum_{s=1}^{n} \int_{\tau + \Delta t}^{t_1} \left[\lambda \left(m_s \ddot{q}_s - \hat{Q}_s \right) \delta \ddot{q}_s + \hat{V}_s \delta q_s \right] dt. \tag{13}$$

When $t \in [\tau, \tau + \Delta t]$ the following equation is true:

$$\sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} [V_s \delta q_s + \lambda (m_s \ddot{q}_s - Q_s) \delta \ddot{q}_s] dt = \\ = \sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} [\delta V_s \delta q_s + \lambda (m_s \delta \ddot{q}_s - \delta Q_s) \delta \ddot{q}_s] dt + \\ + \sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} \left[\hat{V}_s \delta q_s + \lambda \left(m_s \ddot{\hat{q}}_s - \hat{Q}_s \right) \delta \ddot{q}_s \right] dt, \tag{14}$$

where $\delta V_s = V_s - \hat{V}_s$. Because by (1) $m_s \ddot{\hat{q}}_s - \hat{Q}_s = 0$, we get

$$\Delta J_1 = \sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} [\delta V_s \delta q_s - \lambda \delta Q_s \delta \ddot{q}_s] dt + \\ + \sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} \lambda m_s \delta \ddot{q}_s^2 dt + \sum_{s=1}^{n} \int_{\tau}^{t_1} \hat{V}_s \delta q_s dt + \\ + \sum_{s=1}^{n} \int_{\tau + \Delta t}^{t_1} \left[\lambda \left(m_s \ddot{q}_s - \hat{Q}_s \right) \delta \ddot{q}_s \right] dt. \tag{15}$$

Then since $\lambda \left(m_s \left(\ddot{\hat{q}}_s + \delta \ddot{q}_s \right) - \hat{Q}_s \right) \delta \ddot{q}_s = \lambda m_s \delta \ddot{q}_s^2 \geq 0$, $\hat{V}_s \delta q_s \geq 0$, and the integrand F is a positively defined function, to fulfill (7) is enough that

$$\sum_{s=1}^{n} \int_{\tau}^{\tau + \Delta t} \left[\left(V_s - \hat{V}_s \right) \delta q_s - \lambda \left(Q_s - \hat{Q}_s \right) \delta \ddot{q}_s \right] dt \geq 0. \tag{16}$$

Because according to the Gauss principle the trajectories and velocities don't be varied, then the integration by parts leads to the following expression:

$$\sum_{s=1}^{n}\int_{\tau}^{\tau+\Delta t}\left[\frac{d^2\hat{Q}_s}{dt^2}-\lambda^{-1}\hat{V}_s\right]\delta q_s dt \geq \geq \sum_{s=1}^{n}\int_{\tau}^{\tau+\Delta t}\left[\frac{d^2 Q_s}{dt^2}-\lambda^{-1}V_s\right]\delta q_s dt. \quad (17)$$

So

$$\sum_{s=1}^{n}\frac{d^2\left(\hat{Q}_s\left(\hat{\mathbf{q}},\dot{\hat{\mathbf{q}}}\right)\right)}{dt^2}=\lambda^{-1}\sum_{s=1}^{n}\hat{V}_s\left(\hat{\mathbf{q}}\right),$$
$$t=\tau,\ \hat{\mathbf{q}}=\hat{\mathbf{q}}(\tau),\ \dot{\hat{\mathbf{q}}}=\dot{\hat{\mathbf{q}}}(\tau), \quad (18)$$
$$t=\tau+\Delta t,\ \hat{\mathbf{q}}=\hat{\mathbf{q}}(\tau+\Delta t),\ \dot{\hat{\mathbf{q}}}=\dot{\hat{\mathbf{q}}}(\tau+\Delta t).$$

The developing of this equation should be performed separately for each specific case of synthesis problem.

4 The results of mathematical simulation

Let dynamic system (3) has the form:

$$\ddot{q}=Q,$$
$$t_0=0, q(t_0)=0,\ \dot{q}(t_0)=0. \quad (19)$$

It is required to synthesize the law of optimal control of the dynamic system (19), transferring it from initial state to the phase space point $(0,0)$ fulfilling the condition of minimum of the objective functional

$$J=\int_{t_0}^{t_1}dt\to\min; \quad (20)$$

We take $t_1=3$ s.

In accordance with (18)

$$q^{IV}=0, \quad (21)$$

where from

$$Q=\frac{6A^2(q-D)+\left(6ABC-2B^3\right)}{2A(\dot{q}-C)-\left(2B^2-6AC\right)}, \quad (22)$$

where

$$A=-t_1^{-3}\left[12\left(q(t_1)-q(0)-\dot{q}(0)t_1\right)-6t_1\left(\dot{q}(t_1)-\dot{q}(0)\right)\right];$$
$$B=t_1^{-2}\left[6\left(q(t_1)-q(0)-\dot{q}(0)t_1\right)-2t_1\left(\dot{q}(t_1)-\dot{q}(0)\right)\right]; \quad (23)$$
$$C=\dot{q}(0);\ \ D=q(0).$$

The estimation of efficiency of the suggested solution is performed on the basis of comparison with the quasi-optimal law of "soft" terminal control [7, 8]:

$$Q=\frac{12\left(q(t_1)-q(t)\right)}{\left(t_1-t\right)^2}-\frac{6\dot{q}(t_1)-6\dot{q}(t)}{t_1-t}. \quad (24)$$

The results of the mathematical simulation are shown in the Fig, 1, where the number 1 denotes the phase trajectory of the system (19) with the right part (22), and the number 2 denotes the phase trajectory of the system (19) with the right part (24).

Fig, 2 presents the structure of the controlling generalized forces. There are the following notations: 1 – the control (22), 2 – the control (24). It can be seen that singularity (24) at the end time leads to a sharp increase of control force in opposite to (22).

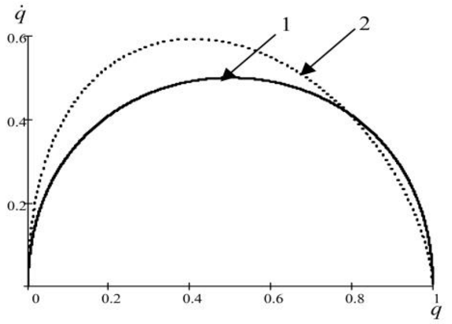

Figure 1. Phase portrait.

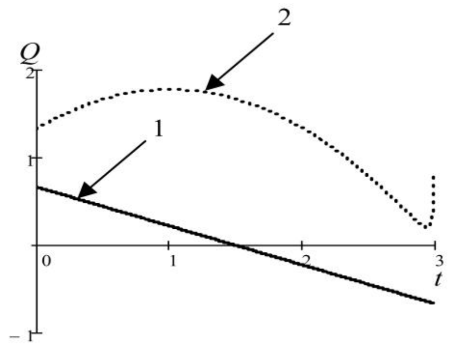

Figure 2. Structure of control.

Considered the other terminal control problem. Let a dynamical system (19) expression (18) takes the form

$$\frac{d^2}{\partial t^2}\frac{\partial^2 G}{\partial\ddot{q}}-\lambda^{-1}q=0,$$
$$t=0,\ q=-1,\ \dot{q}=-3,$$
$$t=t_1,\ q=2,\ \dot{q}=0; \quad (25)$$
$$J=\int_0^{t_1}q^2\to\min.$$

Then

$$q^{IV}+\lambda^{-1}q=0. \quad (26)$$

The solution of this problem can be prove by function of A.N. Krylov

$$q=\sum_{i=1}^{4}A_iK_i(\beta t)=A_1K_1(\beta t)+A_2K_2(\beta t)+ + A_3K_3(\beta t)+A_4K_4(\beta t), \quad (27)$$

where $\beta=\sqrt[4]{\dfrac{\lambda^{-1}}{4}}$, a

$$
\begin{aligned}
&K_1(\beta t) = ch\beta t \cos \beta t,\\
&K_2(\beta t) = \frac{1}{2}\left(ch\beta t \sin \beta t + sh\beta t \cos \beta t\right),\\
&K_3(\beta t) = \frac{1}{2} sh\beta t \sin \beta t,\\
&K_4(\beta t) = \frac{1}{4}\left(ch\beta t \sin \beta t - sh\beta t \cos \beta t\right),
\end{aligned}
\tag{28}
$$

are Krylov's functions.
The control equation is

$$
\begin{aligned}
&Q = \sum_{i=1}^{4} A_i \ddot{K}_i(\beta t) = -4\beta_s^2 A_1 K_3(\beta_s t) -\\
&-4\beta_s A_2 K_4(\beta_s t)\beta_s^2 A_3 K_1(\beta_s t) + \beta_s^2 A_4 K_2(\beta_s t),
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
&\ddot{K}_1(\beta t) = -4\beta^2 K_3(\beta t); \quad \ddot{K}_2 = -4\beta^2 K_4(\beta t);\\
&\ddot{K}_3(\beta t) = \beta^2 K_1(\beta t); \qquad \ddot{K}_4 = \beta^2 K_2(\beta t).
\end{aligned}
\tag{30}
$$

The arbitrary constant determinate by boundary conditions :

$$
\begin{aligned}
&t = 0, \quad q^0 = A_1, \quad \dot{q}^0 = A_2\beta;\\
&t = t_1,\\
&q^* - A_1 K_1(\beta t_1) - A_2 K_2(\beta t_1) =\\
&= A_3 K_3(\beta t_1) + A_4 K_4(\beta t_1),\\
&\dot{q}^* + A_1 4\beta K_4(\beta t_1) - A_2 \beta K_1(\beta t_1) =\\
&= A_3 \beta K_2(\beta t_1) + A_4 \beta K_3(\beta t_1).
\end{aligned}
\tag{31}
$$

The results of modeling are illustrated on Fig. 3. The solution based on the Pontryagin's maximum principle allow getting these results. This confirms the validity of the developed method.

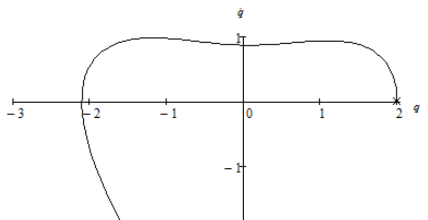

Figure 3. The phase portrait

5 Conclusion

The solution (22) unlike the known solution (24) [7] did not contain singularities at the finite time. As a result, the discontinuity of the control generalized force at the final time is absent. This allows to use it in practice without additional transformation.

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