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Vertex Combination Method for Heat Transfer Analysis of Structures with Uncertain Parameters

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Abstract. Uncertainty is wide-spreads in practical projects which can be solved by interval method. Regarding the temperature field prediction of structures with uncertain parameters, a vertex combination method is presented on the basis of the sampling according to the vertex combinations of interval parameters and monotonic relationship of objective function which can effectively reduce the number of boundary combined samples. A numerical example of heat transfer problems of a panel without thermal resource is used to verify the effectiveness of the proposed method.

1 Introduction

Probability analysis method and non-probability analysis method are the main methods to deal with uncertainty problem. Among them, probability analysis method has the most effectively to express the effect of uncertain parameters of system uncertainty degree quantitatively when it has sufficient sample probability distribution information [1, 2]. However, the exact sample probability distribution information sometimes is difficult to obtain and interval probability model was proposed by domestic and foreign scholars based on interval analysis theory [3, 4], in which does not require exact probability distribution information of uncertain parameters. It only needs to know the uncertain variables in the upper and lower bounds and is easy to utilize and has been widely applied in practice [5-9]. Regarding the temperature field prediction of structures with uncertain parameters, a vertex combination method is presented on the basis of the sampling theory. The system response range can be obtained by the vertex combination method which can effectively reduce the number of boundary combined samples. Finally, a numerical example of heat transfer problems of a panel without thermal resource is used to verify the effectiveness of the proposed method.

2 Vertex combination method

For the steady state heat transfer problem, by using the finite element method the discrete equation of node temperature can be expressed as

$$\mathbf{KT} = \mathbf{F} \tag{1}$$

where $\mathbf{T}$ is node temperature vector; $\mathbf{K}$ is the thermal stiffness matrix; $\mathbf{F}$ is the equivalent thermal load vector.

In practical engineering, it is difficult and expensive to access the parameter probability density function because it needs adequate trial numbers to construct the uncertainty model. However, it gets easier to obtain the roughly range of uncertain parameters compared with the probability density function by using less uncertain information. In this case, we assume that the material properties, external loads and boundary conditions of the heat transfer model can be expressed as

$$\begin{aligned} \boldsymbol{\alpha}^I &= \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right] \\ &= \left(\alpha_i^I\right)_n \\ &= \left([\underline{\alpha}_i, \overline{\alpha}_i]\right)_n \\ &= \left(\alpha_i^c + \Delta\alpha_i\delta\right)_n \\ &= \boldsymbol{\alpha}^{\mathbf{c}} + \Delta\boldsymbol{\alpha}\delta \qquad i = 1, 2, \cdots, n \end{aligned} \tag{2}$$

where $\underline{\alpha}_i, \overline{\alpha}_i$ is the lower and upper bounds; $\alpha_i^c = (\overline{\alpha}_i + \underline{\alpha}_i)/2$ and $\Delta\alpha_i = (\overline{\alpha}_i - \underline{\alpha}_i)/2$ are called the interval midpoint and interval radius respectively; interval parameter $\delta = [-1,1]$ can be expressed simply by δ ; n is the number of uncertain parameters.

The interval parameters of this group will result in the uncertainty of the thermal stiffness matrix $\mathbf{K}$ and the thermal load vector $\mathbf{F}$ in Eq. (1). The steady state heat transfer equation with interval parameters can be rewritten as

$$\mathbf{K}(\boldsymbol{\alpha}^I)\mathbf{T}(\boldsymbol{\alpha}^I) = \mathbf{F}(\boldsymbol{\alpha}^I) \tag{3}$$

Vertex combination method is an interval analysis method to determine the upper and lower bounds of the system response by the boundary combination of interval variables according to the equation

$$\mathbf{K}^I\mathbf{T}^I = \mathbf{F}^I \tag{4}$$

where $\mathbf{K}^I = (k_{ij}^I)_{n\times n}$, $\mathbf{F}^I = (f_i^I)_n$.

Based the interval theory, the solution sets of interval linear equations Eq. (4) is convex and the extreme values are located in interval variables boundary (vertex point). Therefore, the lower bound $\underline{\mathbf{T}}$ and upper bound $\overline{\mathbf{T}}$ of the interval temperature response $\mathbf{T}^I$ can be obtained by the following expressions

$$\begin{aligned}\underline{\mathbf{T}} = (\underline{T}_j) &= \min_{\substack{1\le m\le 2^{n\times n}\\ 1\le q\le 2^n}} (T_j^{mq}) \\ &= \min_{\substack{1\le m\le 2^{n\times n}\\ 1\le q\le 2^n}} \left(\mathbf{K}_m^{-1}\mathbf{F}_q\right) \\ \overline{\mathbf{T}} = (\overline{T}_j) &= \max_{\substack{1\le m\le 2^{n\times n}\\ 1\le q\le 2^n}} (T_j^{mq}) \\ &= \max_{\substack{1\le m\le 2^{n\times n}\\ 1\le q\le 2^n}} \left(\mathbf{K}_m^{-1}\mathbf{F}_q\right)\end{aligned} \tag{5}$$

where $\mathbf{K}_m = \left(\hat{k}_{ij}^m\right)$ is the vertex thermal stiffness matrix under the condition of $\hat{k}_{ij}^m = \underline{k}_{ij}$ or $\hat{k}_{ij}^m = \overline{k}_{ij}$ $i,j = 1,2,...,n; m = 1,2,...,2^{n\times n}$; $\mathbf{F}_q = \left(\hat{f}_i^q\right)$ is the vertex equivalent load vector under the condition of $\hat{f}_i^q = \underline{f}_i$ or $\hat{f}_i^q = \overline{f}_i$ $i = 1,2,...,n; q = 1,2,...,2^n$.

The traditional vertex combination method is to select the minimum and maximum values of the response of the system by the combination of all the elements in the interval matrix and the load vector. As can be seen from Eq. (3), interval matrix and vector of each element changes depending on the parameter vector. That is the interval matrix and vector elements are related to each other and usually not possible at the same time to get the upper or lower bounds. Therefore, the results obtained by using the vertex combination method based on the interval linear equations Eq. (4) are conservative. On this basis, we propose a vertex combination method based on interval parameters.

A n-dimensional hypercube is a uncertain space by combining with n independent interval parameter $\alpha_i^I = [\underline{\alpha}_i, \overline{\alpha}_i]$. When the temperature response on the interval input parameters satisfy a certain relation, the extreme temperature response has been reached in the vertices of the hypercube. Therefore, the lower bound $\underline{\mathbf{T}}$ and upper bound $\overline{\mathbf{T}}$ of the interval temperature response $\mathbf{T}^I$ can be obtained by the following expressions

$$\begin{aligned}\underline{\mathbf{T}} &= (\underline{T}_j) \\ &= \min_{1\le m\le 2^n} (T_j^m) \\ &= \min_{1\le m\le 2^n} \left(\mathbf{K}_m^{-1}\mathbf{F}_m\right) \\ \overline{\mathbf{T}} &= (\overline{T}_j) \\ &= \max_{1\le m\le 2^n} (T_j^m) \\ &= \max_{1\le m\le 2^n} \left(\mathbf{K}_m^{-1}\mathbf{F}_m\right)\end{aligned} \tag{6}$$

where $\mathbf{K}_m = \mathbf{K}(\alpha_1^m,...,\alpha_n^m)$ is the vertex thermal stiffness matrix under the condition of $\alpha_i^m = \underline{\alpha}_i$ or $\alpha_i^m = \overline{\alpha}_i$ $i = 1,2,...,n$; $\mathbf{F}_m = \mathbf{F}(\alpha_1^m,...,\alpha_n^m)$ is the vertex equivalent load vector under the condition of $\alpha_i^m = \underline{\alpha}_i$ or $\alpha_i^m = \overline{\alpha}_i$ $i = 1,2,...,n$.

There is 2^n calculation times for n independent interval parameter vertex combinations. Compared with combinations of vertices of the interval linear equation shown in Eq. (5), it has less calculation and more accurate. However, when the number of uncertain parameters n is relatively large, the amount of calculation of the combination of the vertex can not be ignored.

Actually, by using the central difference method, we can obtain the first-order derivative values of a nodal temperature response T_j on a range of input parameters α_i^I , and then judge function monotone from the temperature response interval parameters.

$$\frac{\partial T_j(\alpha_i^c)}{\partial \alpha_i^I} \approx \frac{T_j(\alpha_i^c + \delta^*) - T_j(\alpha_i^c - \delta^*)}{2\delta^*} \tag{7}$$

where δ^* represents a small change in the midpoint of the parameter α_i^c .

We assume the node temperature response T_j is monotonically increasing with respect to the variable interval $\alpha_1^I, \alpha_2^I, ..., \alpha_k^I$ and monotonically decreasing with respect to the variable interval $\alpha_{k+1}^I, \alpha_{k+2}^I, ..., \alpha_n^I$. Then in solving the the lower and upper bounds ($\underline{T}_j$ and $\overline{T}_j$) of temperature response, we just consider two interval parameters combinations as following

$$\begin{aligned}&\mathbf{K}(\underline{\alpha}_1,...,\underline{\alpha}_k,\overline{\alpha}_{k+1},...,\overline{\alpha}_n)\mathbf{T} \\ &= \mathbf{F}(\underline{\alpha}_1,...,\underline{\alpha}_k,\overline{\alpha}_{k+1},...,\overline{\alpha}_n) \\ &\mathbf{K}(\overline{\alpha}_1,...,\overline{\alpha}_k,\underline{\alpha}_{k+1},...,\underline{\alpha}_n)\mathbf{T} \\ &= \mathbf{F}(\overline{\alpha}_1,...,\overline{\alpha}_k,\underline{\alpha}_{k+1},...,\underline{\alpha}_n)\end{aligned} \tag{8}$$

Compared with the direct combination of the interval parameters shown in Eq. (6), it can greatly reduce the workload by the prior judgment of the monotonic relationship between the temperature response and the parameters.

3 Simulating Process of VCM

The process of solving the upper and lower bounds of the interval temperature response with the method of vertex combination is shown in Fig. 1.It need to point out that the effectiveness of the vertex combination method is established on the basis of the monotonic relationship between the temperature response and the uncertain parameters. For many complex problems in practical engineering, the monotone of function is not strictly satisfied in result that the calculation error of vertex combination method will be larger, and even error results. In addition, when the interval uncertain parameters are more, a large number of combinations of parameter boundaries will lead to the decrease of computational efficiency. From the point of view of computational accuracy and computational cost, the vertex combination method presented in this section is more suitable for uncertain parameters of uncertain parameters.

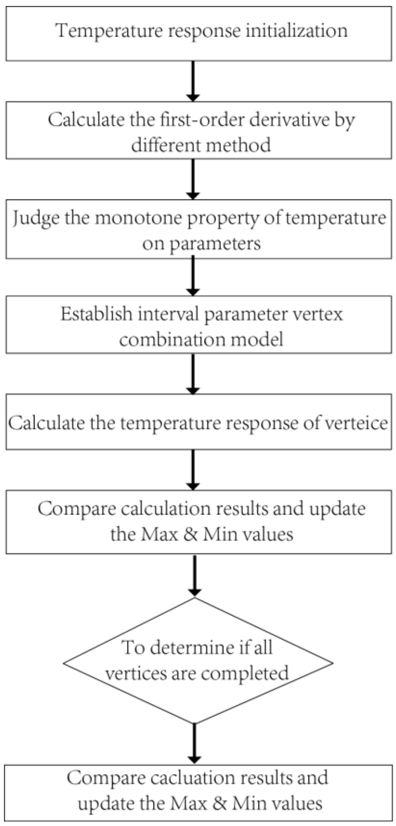

Figure 1. Numerical simulating flow chart of vertex combination method

4 Numerical example

As shown in Fig. 2, the thickness and inner heat source of the flat plate were recorded as 2δ and Q respectively. k is its thermal conductivity, the convection heat transfer between left and right bounds and fluid with temperature T_e. Surface heat transfer coefficient was expressed by h. We will discuss how to determine the temperature response of the flat structure at any position.

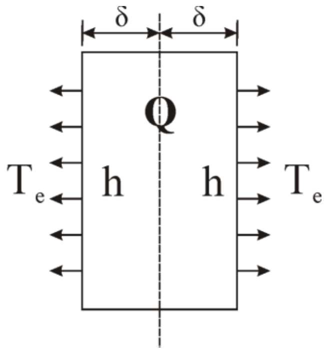

Figure 2. Model of flat wall heat conduction with heat source

The thermal conduction can be expressed by

$$\begin{aligned} & k\frac{\partial^2 T}{\partial x^2} + Q = 0 \\ & x = 0 \quad \frac{\partial T}{\partial x} = 0 \\ & x = \delta \quad -k\frac{\partial T}{\partial x} = h(T - T_e) \end{aligned} \tag{9}$$

We can obtain the solution of the temperature response by using the integral method as following

$$T(x) = \frac{Q}{2k}(\delta^2 - x^2) + \frac{Q\delta}{h} + T_e \tag{10}$$

Assuming the input parameters of the heat transfer problem is in the range of certain changes, which k $\in$ [20,30] W/(m • K), Te $\in$ [10,15] °C, h $\in$ [2000,2500] W/(m^2·K). In addition to the material properties, loads and boundary conditions, this example also consider the uncertainty interval structure dimensions of $\delta \in [5,7]$ cm. The upper and lower bounds of the temperature response at each position are calculated by using the Monte Carlo simulation and the vertex combination sampling methods of the vertex combination in this section, as shown in Table 1.

Table 1. Upper and lower bounds of temperature response along thickness

Location x (cm)	Bounds	Analytical solution (°C)	Monte Carlo Method (10^5 samples)	vertex combination method Method (32 samples)
			Value (°C)	Value (°C)
0	lower	84	84	84
	upper	267	267	267
1	lower	82	82	82
	upper	263	263	263
2	lower	76	76	76
	upper	251	251	251
3	lower	66	66	66
	upper	231	231	231
4	lower	52	52	52
	upper	203	203	203
5	lower	34	34	34
	upper	167	167	167

5 Summary

In view of the steady state heat transfer problem, a numerical discrete model with uncertain parameters is established, and the interval uncertainty analysis technique is proposed based on the sampling principle. Vertex combination method is easy to implement, and the calculation results are accurate. But a large number of vertices will lead to the decrease of computational efficiency, which limits the generalization of the problem to the high dimension complex problem.

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