

# Stochastic Subspace Method for Experimental Modal Analysis

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**Abstract.** The formula of stochastic subspace identification method is deduced in details and the program is written out. The two methods are verified by a vibration test on a 5-floor rigid frame model. In this test the gauss white noise generated from a shaker table to simulate the ambient vibration on the model, and the response signals are measured. Next, the response data of experiment are processed by auto-cross spectrum density method and stochastic subspace identification method respectively, the two methods are verified by comparing with the theory result. and bearing out the superiority of stochastic subspace identification method compared to auto-cross spectrum density method.

## 1 Introduction

Parameter identification is the emphasis of the structural healthy test, and stochastic subspace is an effective structural modal parameter identification method, its advantages are as follows:

**Parameterization:** From data fits, obviously, system identification arithmetic requires stated specific parameters. Only parameter required the rank number of the system in the subspace identification arithmetic as the application of state space model. But classical arithmetic needs a lot of work to confirm alleged canonical model, as the model of minor parameter number.

**Convergence:** Subspace identification arithmetic converges fast. QR decomposition and SVD decomposition used in the method are not required iterative operations, so its convergence is faster than classical arithmetic and usually the convergence question doesn't exist in this method. This mathematics comes from value linear algebra, so value stability is guaranteed exactly. So convergence and value stability need not be considered.

**Model curtailment:** As operating system analysis, as soon as possible, low rank number is always wished to be obtained. In the subspace identification, Curtailment model can obtain from I/O data, it doesn't need to calculate the model with high rank.

## 2 System stochastic state space model [1]

The equation of systemic vibration can be expressed as:

$$M\ddot{U}(t) + C\dot{U}(t) + KU(t) = F(t) = Bu(t) \quad (1)$$

where,  $M, C, K$  are mass matrix, damping matrix and stiffness matrix respectively;  $U(t)$  is structure displacement vector,  $F(t)$  is force vector;  $B$  is place vector;  $u(t)$  is function of time.

Assuming the input is stochastic white noise (gauss distribution). The state equation can be written in time discretization form as:

$$\begin{cases} x_{k+1} = Ax_k + w_k \\ y_k = Cx_k + v_k \end{cases} \quad (2)$$

$$E \left[ \begin{pmatrix} w_p \\ v_q \end{pmatrix} \begin{pmatrix} w_p^T & v_q^T \end{pmatrix} \right] = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \delta_{pq} \quad (3)$$

where,  $y_k \in R^{l \times 1}$  is output vector of  $k$  sampling step ( $\Delta t$ ) at the 1st measuring point,  $x_k \in R^{n \times 1}$  is system state vector;  $n$  is system rank,  $A \in R^{n \times n}$  is state matrix,  $C \in R^{l \times n}$  is output matrix;  $E$  is the mathematics expectation,  $\delta_{pq}$  is Kronecker function,  $w_k \in R^{n \times 1}$  is process noise;  $v_k \in R^{l \times 1}$  is measuring noise;  $w_k, v_k$  are white noise, and  $E(w_k) = 0, E(v_k) = 0$ .

## 3 Stochastic state subspace parameter identification theory [2-4]

In ambient vibration tests, sometimes more measuring points are required, but the number of sensors is limited,

the procedure is divided into several steps. In order to unify each measuring step, the reference measuring points are important and must be selected previously. The

locations of sensors on the reference points must be fixed and data must be recorded continuous. Defining the matrix Hankel block as:

$$H^{ref} \equiv \frac{1}{\sqrt{j}} \begin{pmatrix} y_0^{ref} & y_1^{ref} & \cdots & y_{j-1}^{ref} \\ y_1^{ref} & y_2^{ref} & \cdots & y_j^{ref} \\ \vdots & \vdots & \cdots & \vdots \\ y_{i-1}^{ref} & y_{i-1}^{ref} & \cdots & y_{i+j-2}^{ref} \\ y_i & y_{i+1} & \cdots & y_{i+j-1} \\ y_{i+1} & y_{i+2} & \cdots & y_{i+j} \\ \vdots & \vdots & \cdots & \vdots \\ y_{2i-1} & y_{2i} & \cdots & y_{2i+j-2} \end{pmatrix} \equiv \begin{pmatrix} Y_{0/i-1}^{ref} \\ Y_{0/2i-1} \end{pmatrix} \equiv \begin{pmatrix} Y_p^{ref} \\ Y_f \end{pmatrix} \in R^{(r+l)l \times j} \quad (4)$$

Using QR decomposition to Hankel matrix:

$$H = \begin{bmatrix} Y_{past} \\ Y_{future} \end{bmatrix} = RQ^T = \begin{bmatrix} R_{11} & O \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \quad (5)$$

According to theory of the space projection, the orthogonal projection is expressed by:

$$O_i = \frac{Y_{future}}{Y_{past}} = Y_{future} Y_{past}^T (Y_{past} Y_{past}^T)^{-1} Y_{past} = R_{21} Q \quad (6)$$

where,  $\frac{Y_{future}}{Y_{past}}$  is orthogonal projection of  $Y_{future}$ 's row space on  $Y_{past}$ 's row space;  $(Y_{past} Y_{past}^T)^{-1}$  is Moore-Pseudo inverse matrix of  $Y_{past}$ .

The velocity of the programmer's operation is increased by using QR decomposition.

$$O_i = USV^T = [U_1 \quad U_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \approx U_1 S_1 V_1 \quad (7)$$

where, both  $U \in R^{il \times il}$ ,  $V \in R^{j \times j}$  are orthogonal matrixes, and satisfy  $U^T U = U U^T = I_{il}$ ,  $V^T V = V V^T = I_j$ ,  $S \in R^{il \times j}$  is a diagonal matrix, and row full order in the condition of common noise disturbance, but there are several obvious 'broken points' for the element sizes on the diagonal, the system rank can be obtained from the number of the diagonal elements before the 'broken point'.

SVD is widely applied to solving the problem of main component analysis, for a set of signal from some system structure,  $U_1 S_1 V_1^T$  contains main information of the signal, and  $U_2 S_2 V_2^T$  contains secondary information of the signal, and which is considered as the noise of the signal. If the system isn't polluted by noise,  $S_2$  is equal to zero. But in fact, the noise disturbance is unavoidable, so

$S_2$  is not equal to zero; But  $S_2$  is smaller than  $S_1$ , so  $U_2 S_2 V_2^T$  can be ignored and the aim result of the noise is achieved

According to the subspace system identification theory<sup>[5,6]</sup>, projection matrix  $O_i$  is also equal to the product between generalized observable matrix C and non-steady Kalman filer estimation  $\hat{X}_i$  of the state sequence  $X_i$ :

$$O_i = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \begin{bmatrix} \hat{x}_i & \hat{x}_{i+1} & \cdots & \hat{x}_{i+j-1} \end{bmatrix} = \Gamma_i \hat{X}_i \quad (8)$$

The main purpose of Kalman filter is to obtain future state vector by optimization forecast based on fore-passed state vectors (output data before the time k), system matrix and noise covariance  $\Gamma_i$  is obtained by SVD using Hankel matrix.

$$\Gamma_i = U_1 S_1^{1/2}, \hat{X}_i = S_1^{1/2} V_1^T \quad (9)$$

Generalized observable matrix is calculated by SVD. To obtain systemic generalized observable matrix, SVD is done to projection matrix. The mention is row full order commonly in the condition of noise disturbance, but there are several obvious ‘broken points’ on the element sizes on the cross, the systemic rank number can be obtained from the cross elements’ number before the ‘broken point’. SVD decomposition is widely applied to solving the problem of main component analysis. a set of signal from some system structure, contains main information of the signal, and secondary information of the signal is considered as the noise of the signal. Matrix A and C are obtained by generalized observable matrix: the following equation can be obtained by the equation (7):

$$C = \Gamma_i(1:l,:) \quad (10)$$

$$A = [\Gamma_i(1:il-l,:)]^* \Gamma_i(l+1:il,:) \quad (11)$$

where,  $\Gamma_i(1:l,:)$ —first row block of  $\Gamma_i$ ;

$[\Gamma_i(1:il-l,:)]^*$ —fake inverse matrix of Moore – Penrose of foregoing (i-1) row block of  $\Gamma_i$ ;

$\Gamma_i(l+1:il,:)$ —later (i-1) row block of  $\Gamma_i$ .

Systemic dynamic characteristics are divided by the eigenvalue and eigenvector of system matrix A. The purpose of the paper is only to indentify three mode parameters of the system: nature frequency, damping ratio and mode shape, which are only related to system matrix A and observable matrix C. So it is enough to calculate A and C, and the stochastic subspace is ignored here.

#### 4 Structure mode analysis<sup>[7 ~ 10]</sup> and procedure

Eigenvalue decomposed by System matrix A is as follows:

$$\Lambda = \Psi \Lambda \Psi^{-1} \quad (12)$$

where,  $\Lambda = \text{diag}(\lambda_m) \in R^{n \times n} (m = 1, 2, \dots, n)$ ;  $\lambda_m$  is eigenvalue of discretion system,  $\Psi \in R^{n \times n}$  is systemic eigenvector matrix.

The relation between the eigenvalue and the nature frequency  $\omega$ , damping ratio  $\zeta$  is as follows:

$$\lambda_i^c, (\lambda_i^c)^* = -\zeta_i \omega_i \pm j \omega_i \sqrt{1 - \zeta_i^2} \quad (13)$$

where,  $(\lambda_i^c)^* = \lambda_i^c$  is the conjugate form.

The nature frequency of the system is:

$$\omega_i = |\lambda_i^c|, f_i = \frac{\omega_i}{2\pi} \quad (14)$$

where,  $(\lambda_i^c)^R$  is real part of  $\lambda_i^c$ .

As the mode shape of the measuring point is observed from eigenvector of the system. The mode shape of nature equation can be written as:

$$\Phi = C \Psi \quad (15)$$

The MATLAB procedure chart of stochastic subspace method is as follows (Fig. 1):

For the determination of model order, and for the decomposition of spectrum mateices. In theory, the number of non-zero singular value is rank of spectrum matrix, the order of the model. but in fact, spectrum matrix is obtained by actual test data. More or less it contain noise in actual test, and it result that singular value in the matrix is not really zero. Singular value can produce sudden drop, but fell sharply before the singular value can be thought of as order of the model.

#### 5 Data processing with stochastic subspace method compared with the results of FEM

The natural frequency from finite element analysis<sup>[11-13]</sup> is as follows (Table 1):

**Table 1.** Natural frequency from finite element analysis

Order of the element	1	2	3	4	5	6
Natural frequency (HZ)	10.825	30.091	37.889	54.234	67.953	70.632

The mode shape from finite element analysis is as follows:

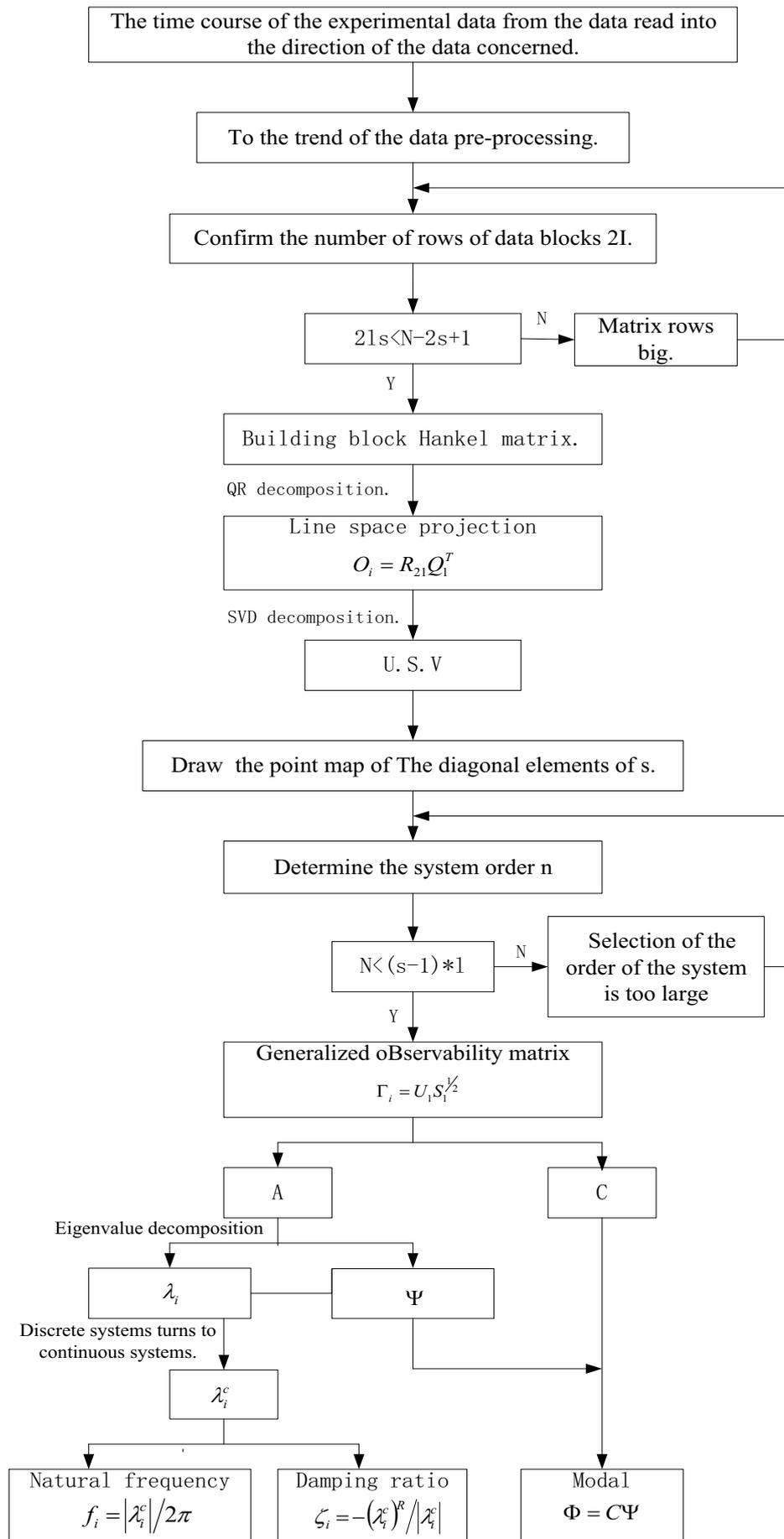
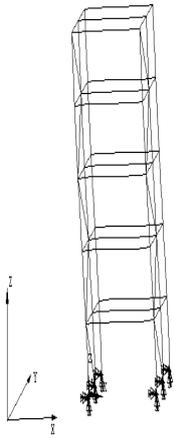
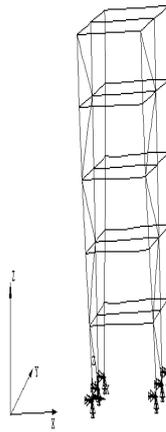


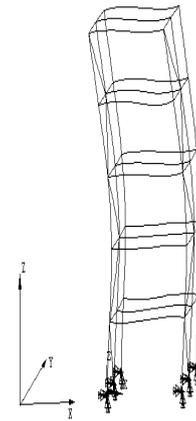
Figure 1. MATLAB procedure flow chart



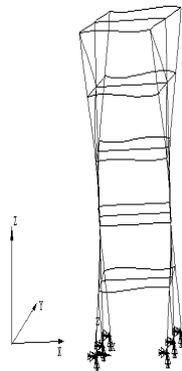
**Figure 2.** The first modal shape



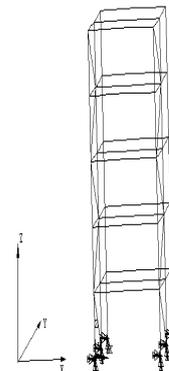
**Figure 3.**The second modal shape



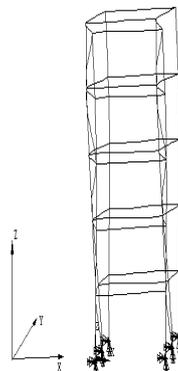
**Figure 4.**The third modal shape



**Figure 5.** The first modal shape



**Figure 6.** The second modal shape



**Figure 7.** The third modal shape

The number of the row block of Hankel Matrix is 200 when the data is processed by the stochastic subspace method. As the calculation capacity is limited, only the data of the first 5000 discrete point is used in x and y directions. The 4 nature frequencies in x direction and the 2 nature frequencies below 100 Hz in y direction, listed in table 1. The first 6 mode shapes by stochastic subspace method list as follows:

1,3 modal shapes are without bracing the x direction (the direction of small united) of a bend, two bends(2nd bends), bending vibration mode 3 plus bracing for the y-direction (the direction of large bending rigidity) of a bending vibration mode, 2、6 vibration mode for the direction of a small twist twisting stiffness, torsional type 2, five vibration mode for the large distortions in the

direction of a torsional stiffness, 4 vibration mode, respectively, for small united of the bending direction of 2 plus a small distortion torsional stiffness of 1 direction , 2 torsional united model for the small plus small distortions in the direction of 3 united of the direction of a torsional type.

It is very close that the results are obtained by the finite element analysis and the stochastic subspace method, and it proves that the stochastic subspace method is reliable.

The test data of Hankel Matrix is processed by the stochastic subspace method, and the modal parameter is identified. Then the results are compared with the finite element analysis, and it shows that the stochastic subspace method is available.

**Table 2.** Result of modal analysis

Order	Finite element method	cross-spectral method			Random subspace (SSI)		
	Natural frequency (Hz)	Natural frequency (Hz)	Frequency error (%)	Damping ratio (%)	Natural frequency (Hz)	Frequency error (%)	Damping ratio (%)
1	10.825	10.90	0.69	0.61	10.9612	1.26	0.6527
2	30.091	29.85	0.80	0.27	29.8193	0.90	0.4354
3	37.889	37.95	0.16	0.34	37.6066	0.75	1.0480
4	54.234	—	—	—	54.6168	0.71	0.4936
5	67.953	65.55	3.54	1.18	65.7421	3.25	1.0483
6	70.632	70.45	0.26	0.19	70.4692	0.23	0.3650

## 6 Conclusions

The identified nature frequencies by self mutual chart method and stochastic subspace method are close to the theory result and identification precision satisfies the request. It shows that the two methods are effective.

The first six modals can be identified by stochastic subspace method, but cannot by self mutual. It shows that the test is successful and stochastic subspace method is more excellent than self mutual chart method. Stochastic subspace method can be used in dense mode identification broadly.

The results obtained by the two identification methods are very close, and it proves that the two methods are reliable. Although self mutual chart method is a traditional graphic method and has some shortcomings. It can identify accurately the first several nature frequencies by data's reasonable adding window and fit. So the identified result by self mutual chart method has great reference value. And the self mutual chart method can be a supplement to the stochastic subspace method for its convenient.

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